# On the Field Theoretic Functional Calculus for the Anharmonic Oscillator. II

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The theory of solution for quantum field functional equations is developed for a suitable testproblem of quantum mechanics. In Sect. 1 the anharmonic oscillator is described in a field theoretic fashion. In Sect. 2 its functional equations are derived and in Sect. 3 these equations are symmetrized due to physical conditions. In Sect. 4, 5 the expansion of the physical functionals into series of base functionals is discussed and a convenient notation for the operator representation is introduced. In Sect. 6 the representation of the functional equation for an expansion into Dyson base functionals is given. In Sect. 7 and 8 functionals are approximated by expansions with only a finite number of terms and the resulting equations are prepared for integration. In Sect. 9, 10 the integration of the resulting equations for $N = 2$ and $N = 4$ is discussed in detail so that one finally obtains eigenvalue equations which contain only integrals to be solved. In the appendices technical details are derived.

In nonlinear spinor theory the dynamical behaviour of elementary particles can be described by functionals of field operators in a Heisenberg representation and corresponding functional equations [1, 2, 3]. To obtain the physical information, it is necessary to solve the functional equations without perturbation theory, i.e. for the strong coupling case. Therefore a theory of solution for these equations has to be developed. As has been discussed in previous papers, the anharmonic oscillator is a suitable test problem for the investigation of strong coupling functional equations [2, 4, 5, 6]. First results about the solution procedure in the one time limit of the functional equation for this simple system have been given [6]. But to obtain a complete functional analogy between the test system and nonlinear spinor theory the investigation of the many time functional equations is required [2,3]. This problem is attacked now in this paper. The general idea for the solution procedure is the use of an expansion of the physical functionals into series of suitably chosen base functionals and to approximate the exact infinite series by series with a finite number of terms. The general theory then requires the proof of convergence for this procedure and the explicit calculation of the approximate functionals. Only the second problem is discussed in this paper. As has been shown in [6] the approximation procedure can be performed either in symmetrical or in unsymmetrical functional operator representations. Although the first possibility seems to be more promising from a general mathematical point of view [6] the second possibility of working with an unsymmetrical representation leads to the usual field theoretic matrix formalism. To maintain the connection with former field theoretic calculations we choose the unsymmetrical representation but we emphasize that the solution procedure discussed here can be applied equally well to the symmetrical representation. Furthermore we want to emphasize that this procedure can be applied without any modification to nonlinear spinor theory i.e. a relativistic field theory. One has still the choice to work in the $q$-representation (corresponding to relativistic scalar fields) or in the $p$-$q$-representation (corresponding to relativistic spinor fields) [2, 6]. For convenience we use the $q$-representation. Again its solution procedure can be applied equally well to the $p$-$q$-representation. We develop the solution procedure for the states of even parity (corresponding to boson states) up to the point where numerical calculations can be performed. The solution procedure for states of odd parity (corresponding to fermion states) runs completely analogous. Numerical values are only given for the lowest approximation in Sect. 9. As for higher approximations the numerical effort increases considerably, we do not try to give for these approximations numerical values too. This will be done in special papers, some of which are already in preparation.

[1] W. HEISENBERG, An Introduction to the Unified theory of Elementary Particles, Wiley and Sons, London 1967.
[2] H. RAMPACHER, H. STUMPF and F. WAGNER, Fortschr. Phys. **13**, 385 [1965].
[3] H. P. DÜRR and F. WAGNER, Nuovo Cim. (X) **46**, 223 [1966].
[4] W. HEISENBERG, Nachr. Gött. Akad. Wiss. **1953**, 111.
[5] H. STUMPF, F. WAGNER and F. WAHL, Z. Naturforschg. **19a**, 1254 [1964].
[6] D. MAISON and H. STUMPF, Z. Naturforschg. **21a**, 1829 [1966].

## 1. Field Theoretic Representation

We consider the anharmonic oscillator as a simple field theoretic model. Therefore we formulate its basic quantum mechanical relations in the way of quantum field theory. The equations of motion are

$$\dot{q}(t) = p(t), \quad \dot{p}(t) = -q^3(t) \tag{1.1}$$

with the commutation relation

$$[p(t), q(t)] = -i\,\mathbf{1}. \tag{1.2}$$

The stationary states of the anharmonic oscillator i.e. the eigenstates of the time translational operator $H$ are a complete system of the corresponding HILBERT space and admit the representation

$$\langle \Psi_\varrho | = \sum_{n=0}^{\infty} \sigma_n(\varrho) \langle F_n | \quad (\varrho = 0, \ldots, \infty) \tag{1.3}$$

with the base vectors

$$\langle F_n | := \langle 0 | q^n(0) \quad (n = 0, \ldots, \infty). \tag{1.4}$$

$\langle 0|$ being the physical ground state normed to unity. As has been proven in [7] the expansion states (1.4) are also complete but not orthogonal. Then it is convenient to introduce a second system of base vectors, called the reciprocal base system, to formulate the HILBERT space norm. This system is defined by

$$\langle F_n | R_k \rangle = \delta_{nk} \tag{1.5}$$

and the eigenstates (1.3) have to admit an equal representation by the expansion

$$|\Psi_\varrho\rangle = \sum_{l=0}^{\infty} \tau_l(\varrho) | R_l \rangle. \tag{1.6}$$

We get the scalar product between the two states according to (1.5) as

$$\langle \Psi_\varrho | \Psi_\varrho \rangle = \sum_{n=0}^{\infty} \sigma_n(\varrho)\, \tau_n(\varrho). \tag{1.7}$$

For the actual construction of the states $|\Psi_\varrho\rangle$ one has to calculate the expansion coefficients either of (1.3) or of (1.6) given by the projections

$$\langle \Psi_\varrho | R_l \rangle = \sigma_l(\varrho) \tag{1.8}$$

or

$$\langle F_n | \Psi_\varrho \rangle = \langle 0 | q^n(0) | \Psi_\varrho \rangle = \tau_n(\varrho). \tag{1.9}$$

The two sets of expansion coefficients are not independent from another. They are related by a system of algebraic equations. To derive them we consider

$$\tau_n^\times(\varrho) = \langle \Psi_\varrho | q^n(0) | 0 \rangle \tag{1.10}$$

which is a consequence of the HERMITEAN property of the $q$-operator. Then with the definition

$$| F_k \rangle := q^k(0) | 0 \rangle \tag{1.11}$$

we obtain from (1.3)

$$\tau_k^\times(\varrho) = \sum_{n=0}^{\infty} \sigma_n(\varrho) \langle F_n | F_k \rangle \tag{1.12}$$

being an infinite system of algebraic equations for the calculation of the expansion (1.3) if the reciprocal expansion (1.6) is known.

[7] F. WAGNER, Thesis, University of Munich 1966.

## 2. Functional Representation

In the preceding section we discussed the field theoretic version of the anharmonic oscillator. Now we turn to the calculational problem, i.e. the calculation of stationary states and norms. In this paper we discuss only states. The states are known if their expansion functions either of the expansion (1.3) or of the expansion (1.6) are explicitly given. To maintain the full analogy to quantum field theory, especially to nonlinear spinor theory, we ignore the possibility of state calculations in the SCHRÖDINGER picture but try to calculate the set of $\tau$-coefficients of (1.6). In order to do this we observe that the $\tau_n(\varrho)$-values are limiting values of the many-time $\tau$-functions.

$$\tau_\varrho(t_1 \ldots t_n) := \langle 0 | T\, q(t_1) \ldots q(t_n) | \Psi_\varrho \rangle \tag{2.1}$$

where $T$ means time ordering. Then we have according to the definition of the $T$-product

$$\tau_n(\varrho) = \lim_{t_1 > \ldots > t_n \to 0} \tau_\varrho(t_1 \ldots t_n). \tag{2.2}$$

Therefore the states $|\Psi_\varrho\rangle$ can also be characterized by the many-time $\tau$-functions (2.1). It is this feature which completes the analogy to relativistic quantum field theory. In the following we therefore try to calculate the many-time functions (2.1) although they contain superfluous information. For their calculation we introduce an auxiliary space, the so-called functional space, where the set of $\tau$-functions is represented by a functional in the following form

$$\mathfrak{T}_\varrho(j) := \sum_{k=1}^{\infty} \int \tau_\varrho(t_1 \ldots t_k)\, F(t_1 \ldots t_k, j)\, \mathrm{d}t_1 \ldots \mathrm{d}t_k. \tag{2.3}$$

The base functionals in the expansion (2.3) are defined by

$$F(t_1 \dots t_k, j) := \frac{i^k}{k!} j(t_1) \dots j(t_k) \tag{2.4}$$

with classical source functions $j(t)$. Observing (2.3), (2.4) and the definition (2.1) the functional (2.3) may also be written as

$$\mathfrak{T}_\varrho(j) = \langle 0 | \, i \, T \exp \int q(t) \, j(t) \, \mathrm{d}t \, | \Psi_\varrho \rangle \,. \tag{2.5}$$

For functionals one is able to define a functional differentiation and a functional integration[8,9]. Especially by the differentiation operation one can derive from the dynamical equations of Section 1 and of (2.5) a functional equation characterizing $\mathfrak{T}_\varrho(j)$. For details of its derivation we refer to Appendix I. One obtains

$$\frac{d^2}{dt^2} \frac{\delta}{\delta j(t)} \mathfrak{T}_\varrho(j) = U\left(j(t), \frac{\delta}{\delta j(t)}\right) \mathfrak{T}_\varrho(j) \tag{2.6}$$

with $$U\left(j(t), \frac{\delta}{\delta j(t)}\right) := \frac{\delta^3}{\delta j(t)^3} + i \, j(t) \,. \tag{2.7}$$

Additionally for stationary physical functionals the necessary subsidiary condition

$$\int j(t) \frac{d}{dt} \frac{\delta}{\delta j(t)} \mathrm{d}t \, \mathfrak{T}_\varrho(j) = - \, i \, \omega_\varrho \, \mathfrak{T}_\varrho(j) \tag{2.8}$$

has to be satisfied[6]. Both equations are the starting point of our calculational process in the following.

### 3. Symmetrized functional equations

As it is our opinion that the problems in quantum field theory can only be solved appropriately by using the methods of integral equations we just change the differential equation (2.6) into an integral equation. It is convenient to introduce normal ordering of the interaction term by adding on both sides of (2.6) a contraction term[2] resulting in

$$\left[\frac{d^2}{dt^2} + 3\,F(0)\right] \frac{\delta}{\delta j(t)} \mathfrak{T}_\varrho(j) = N\left(j(t), \frac{\delta}{\delta j(t)}\right) \mathfrak{T}_\varrho(j) \tag{3.1}$$

with

$$N\left(j(t), \frac{\delta}{\delta j(t)}\right) := U\left(j(t), \frac{\delta}{\delta j(t)}\right) + 3\,F(0) \frac{\delta}{\delta j(t)} \tag{3.2}$$

where $F(0)$ is the vacuum expectation value of $q^2$. Defining the FEYNMAN-GREEN function for (3.1) by

$$\left[\frac{d^2}{dt^2} + 3\,F(0)\right] G(t-t') = \delta(t-t') \tag{3.3}$$

from (3.1) follows by application of $G$ the functional equation

$$\frac{\delta}{\delta j(t)} \mathfrak{T}_\varrho(j) = \int G(t-t') \, N\left(j(t'), \frac{\delta}{\delta j(t')}\right) \mathrm{d}t' \, \mathfrak{T}_\varrho(j). \tag{3.4}$$

But still Eq. (3.4) is not satisfactory because for every $\tau$-function in configurational space there are several separated integral equations, which do not reflect the symmetry properties of these functions. It is necessary to combine these equations in such a way that the resulting equations have only symmetric operators in all arguments. The first approach in this direction was done by MATTHEWS and SALAM[10]. Later on DÜRR and WAGNER[3] demonstrated that the MATTHEWS-SALAM equation is only a special linear combination formed from all possible symmetrical equations in configurational space. Thus it remains open which linear combination is the most advantageous one. A formal prescription has already been given in[2] for the differential form of the functional equation. Here we modify this description for the integral form of the functional equation and show that this equation has the same manifold of solutions as the original Eq. (3.4), provided that the solutions are symmetrical in all arguments. We propose the following equation

$$\int j(t) \frac{\delta}{\delta j(t)} \mathrm{d}t \, \mathfrak{T}_\varrho(j) = \int j(t) \, G(t-t') \, N\left(j(t'), \frac{\delta}{\delta j(t')}\right) \mathrm{d}t \mathrm{d}t' \mathfrak{T}_\varrho(j) \,. \tag{3.5}$$

By means of the calculus developed in the next sections it can easily be seen that (3.5) only implies completely symmetrical operations in configurational space. So (3.5) has the desired properties. To replace (3.4) by (3.5) we only have to show that all solutions of (3.5) are solutions of (3.4) and that also the symmetrical solutions of (3.4) satisfy (3.5). First it is clear that (3.4) has at least the physical solutions, following by direct construction from the Schrödinger amplitudes whose existence is secured. Then each physical solution of (3.4) satisfies (3.5) automatically because (3.5) results from a linear combination of (3.4). Therefore in this direction the transition from (3.4) to (3.5) does not cause any difficulty for the physical solutions. If one supposes on the other hand to have a solution of (3.5) being

[8] Y. V. NOVOZHILOV and A. V. TULUB, The Method of Functionals in the Quantum Theory of Fields, Gordon and Breach, New York 1961.

[9] I. FRIEDRICHS and A. SHAPIRO, Seminar on Integration of Functionals, New York University 1957.

[10] P. T. MATTHEWS and A. SALAM, Proc. Roy. Soc. London **A 221**, 128 [1954].

regular at the origin in functional space, i.e. having a power expansion solution like (2.3) and denotes this solution by $\mathfrak{T}_S(j)$; then one may insert this solution into (3.4). This gives

$$\left[\frac{\delta}{\delta j(t)} - \int G(t-t')\, N\left(j(t'), \frac{\delta}{\delta j(t')}\right) \mathrm{d}t'\right] \mathfrak{T}_S(j) = \mathfrak{Z}(t, j) \tag{3.6}$$

where $\mathfrak{Z}(t, j)$ has to have the form

$$\mathfrak{Z}(t, j) := \sum_{n=1}^{\infty} \int z_n(t, t_1 \dots t_n)\, F(t_1 \dots t_n, j)\, \mathrm{d}t_1 \dots \mathrm{d}t_n \tag{3.7}$$

following from the properties of $\mathfrak{T}_S(j)$ and of the operator on the lefthand side of (3.6). Then by multiplying (3.6) with $j(t)$ and integrating over $t$ from (3.6) (3.5) results and because $\mathfrak{T}_S(j)$ is supposed to be a solution of (3.5) we have

$$\int j(t)\, \mathfrak{Z}(t, j)\, \mathrm{d}t = 0\,. \tag{3.8}$$

Now in our entire calculus we may use an arbitrary but square summable $j(t)$. As (3.8) is valid for any $j(t)$ satisfying this condition we therefore conclude $\mathfrak{Z}(t, j) = 0$. This, however, means that $\mathfrak{T}_S(j)$ simultaneously satisfies (3.5) and (3.4). Thus all solutions of (3.5) admitting a power series expansion like (2.3) are automatically solutions of (3.4) too. Therefore we are allowed to use the symmetrized equation (3.5) instead of (3.4). Naturally the physical solutions of (3.5) still have to satisfy (2.8). But this equation has already the desired symmetrical form so that we do not have to symmetrize it further.

## 4. Functional Expansions

The first point of interest in our functional formulation is the investigation of the possibility of different expansions for $\mathfrak{T}(j)$. These expansions will be important when one looks for the best way of representing a given functional approximately, i.e. by only a finite number of expansion terms, this being inevitable for practical calculations. Then it is a crucial question which expansion is the best one for such a truncated description. Since in the one-time limit a norm of $\mathfrak{T}_\varrho(j)$ exists[7], one should think that in analogy to ordinary analysis the most suitable expansions are those with orthonormalized base functionals; as is well known normalizable functions are optimally approximated by orthonormalized base functions. A step in this direction is the introduction of the base functionals

$$D(t_1 \dots t_n, j) := F(t_1 \dots t_n\, j) \exp\{-\tfrac{1}{2} \int j(\xi)\, F(\xi - \eta)\, j(\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta\} \tag{4.1}$$

which under certain conditions on $F(\xi - \eta)$ are normalizable but not orthogonal. Then we assume $\mathfrak{T}_\varrho(j)$ to have the expansion

$$\mathfrak{T}_\varrho(j) = \sum_{k=1}^{\infty} \int \varphi_\varrho(t_1 \dots t_k)\, D(t_1 \dots t_k\, j)\, \mathrm{d}t_1 \dots \mathrm{d}t_k\,. \tag{4.2}$$

In order to calculate the set of $\tau$-functions Dyson was the first to introduce this set of expansion functionals. Therefore we call the base functionals (4.1) the Dyson base functionals. There are still other expansion functionals possible to define e.g. the famous orthonormalized Hermitean functionals[9]. About their application for the present problem another paper is in preparation [11] so that we do not discuss them here.

In using different representations of $\mathfrak{T}_\varrho(j)$ it is desirable to rewrite the expansion coefficients of one representation in terms of those of another representation. According to functional analysis this should be done by the formation of scalar products between the different base functionals, i.e. in this case by functional integration[9]. From historical reasons in quantum field theory another procedure is used, namely the comparison of equal power coefficients in $j(t)$, because it is assumed that all physical functionals allow a power series expansion like (2.3). In practice such a power series comparison can be achieved by use of the formula

$$\frac{1}{i^l} \frac{\delta^l}{\delta j(t_1') \dots \delta(t_l')} F(t_1 \dots t_k, j) \Big|_{j=0} = \delta_{lk} \frac{1}{k!} P \sum_{\lambda_1 \dots \lambda_k = 1}^{k} \delta(t_1' - t_{\lambda_1}) \dots \delta(t_k' - t_{\lambda_k})\,. \tag{4.3}$$

This equation looks like an orthonormality relation and we use this analogy to formulate all functional operations by the notation of functional analysis, because this notation preserves the base invariant description of operator relations as will be shown in the following.

[11] D. Maison, Thesis, University of Munich, in preparation.

Therefore we replace formally the operator on the left side of (4.3) by a functional scalar product

$$\int R(t_1' \ldots t_l', j)\, F(t_1 \ldots t_k, j)\, \mathrm{d}j := \frac{1}{i^l} \frac{\delta^l}{\delta j(t_1') \ldots \delta j(t_l')} F(t_1 \ldots t_k, j)\big|_{j=0}. \tag{4.4}$$

As the functionals $F(t_1 \ldots t_n, j)$ are neither normalizable nor orthogonal the $R$-set cannot be identical with the $F$-set, but has to be a rather pathological functional set, if one would try to construct it explicitly[12]. This construction is not necessary because in the following we always use the $R$-set in connection with the symbol of functional integration, i.e. only the formula (4.3). So we have no trouble with the existence problem of the $R$-set. But we emphasize, that by a more suitable choice of base functionals the formation of scalar products in functional space becomes a well defined operation[9]. Keeping this in mind we believe that the substitution (4.4) elucidates the mathematical meaning of our operations. Applying now (4.3) respectively (4.4) on $\mathfrak{T}_\varrho(j)$ we then get

$$\int R(t_1 \ldots t_l, j)\, \mathfrak{T}_\varrho(j)\, \mathrm{d}j = \tau_\varrho(t_1 \ldots t_l) \tag{4.5}$$

and
$$\tau_\varrho(t_1 \ldots t_l) = \sum_{k=1}^{\infty} \int \varphi_\varrho(t_1' \ldots t_k') \int R(t_1 \ldots t_l, j)\, D(t_1' \ldots t_k', j)\, \mathrm{d}t_1' \ldots \mathrm{d}t_k'\, \mathrm{d}j \tag{4.6}$$

as the transformation rule between the two sets of expansion coefficients of (2.3) and (4.2). (4.6) becomes the WICK rule if one identifies $F(\xi - \eta)$ with the two point function $\langle 0 | T q(\xi) q(\eta) | 0 \rangle$ i.e. we reproduce by this operation field theoretic results. By the same arguments as used in the foregoing, formally a reciprocal set for the DYSON base functionals can be constructed from the $R$-set. Defining it formally by

$$S(t_1 \ldots t_n, j) := R(t_1 \ldots t_n, j) \exp\{\tfrac{1}{2} \int j(\xi)\, F(\xi - \eta)\, j(\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta\} \tag{4.7}$$

one easily verifies by observing (4.3) and (4.1)

$$\int S(t_1' \ldots t_l', j)\, D(t_1 \ldots t_k, j)\, \mathrm{d}j = \delta_{kl} \frac{1}{k!} P \sum_{\lambda_1 \ldots \lambda_k = 1}^{k} \delta(t_1' - t_{\lambda_1}) \ldots \delta(t_k' - t_{\lambda_k}) \tag{4.8}$$

and from this follows

$$\int S(t_1 \ldots t_l, j)\, \mathfrak{T}_\varrho(j)\, \mathrm{d}j = \varphi_\varrho(t_1 \ldots t_l) \tag{4.9}$$

and
$$\varphi_\varrho(t_1 \ldots t_l) = \sum_{k=1}^{\infty} \int \tau_\varrho(t_1' \ldots t_k') \int S(t_1 \ldots t_l, j)\, F(t_1' \ldots t'_k, j)\, \mathrm{d}t_1' \ldots \mathrm{d}t_k'\, \mathrm{d}j \tag{4.10}$$

as the inverse transformation rule to (4.6). Also this inversion of the WICK rule is known in field theory.

## 5. Functional Operator Representation

To make plain the meaning of the different expansions in Sect. 4 we give in this section a systematic treatment of functional operators. We have already used functional operators in differential form in the preceding sections. But for a systematic treatment it is convenient to change all operators in integral operators. In order to do this first of all we consider a general set of base functionals $B(t_1 \ldots t_n, j)$ and its reciprocal set $T(t_1 \ldots t_k, j)$ satisfying the relations

$$\int T(t_1 \ldots t_n\, j)\, B(t_1' \ldots t_k', j)\, \mathrm{d}j = \delta_{nk} \frac{1}{n!} P \sum_{\lambda_1 \ldots \lambda_n = 1}^{n} \delta(t_1' - t_{\lambda_1}) \ldots \delta(t_n' - t_{\lambda_n}). \tag{5.1}$$

Then we define the integral representation of an operator in the functional space of the $B$- and $T$-sets by

$$O(j, j') = \sum_{k,i=1}^{\infty} \int B(\xi_1 \ldots \xi_k, j)\, O_i^k(\xi_1 \ldots \xi_k, \eta_1 \ldots \eta_i)\, T(\eta_1 \ldots \eta_i, j')\, \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_k\, \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_i \tag{5.2}$$

[12] In ordinary function space the set of $x^n/n!$ ($n = 0, 1, \ldots, \infty$) corresponds to the $F$-set. Its reciprocal system $r_k(x)$ is given by $\delta^{(k)}(x)$ i.e. by distributions.

with

$$O_i^k(\xi_1\ldots\xi_k,\eta_1\ldots\eta_i):=\int T(\xi_1\ldots\xi_k,j)\,O(j,j')\,B(\eta_1\ldots\eta_i,j')\,\mathrm{d}j\,\mathrm{d}j'. \tag{5.3}$$

The unity operator in this space is given by

$$\delta(j,j')=\sum_{k=1}^{\infty}\int B(\xi_1\ldots\xi_k,j)\frac{1}{k!}P\sum_{\lambda_1\ldots\lambda_k=1}^{k}\delta(\xi_1-\eta_{\lambda_1})\ldots\delta(\xi_k-\eta_{\lambda_k})\,T(\eta_1\ldots\eta_k,j')\,\mathrm{d}\xi_1\ldots\mathrm{d}\xi_k\,\mathrm{d}\eta_1\ldots\mathrm{d}\eta_k \tag{5.4}$$

and for the state functionals we assume the expansion

$$\mathfrak{T}_\varrho(j)=\sum_{k=1}^{\infty}\int b_\varrho(\xi_1\ldots\xi_k)\,B(\xi_1\ldots\xi_k,j)\,\mathrm{d}\xi_1\ldots\mathrm{d}\xi_k. \tag{5.5}$$

For this representation the following multiplication rules are valid

$$\int O(j,j'')\,P(j'',j)\,\mathrm{d}j''=Q(j,j'), \tag{5.6}$$

$$\int O(j,j')\,\mathfrak{T}(j')\,\mathrm{d}j'=\mathfrak{R}(j), \tag{5.7}$$

$$\int \delta(j,j')\,\mathfrak{T}(j')\,\mathrm{d}j'=\mathfrak{T}(j). \tag{5.8}$$

(5.6) means operator multiplication, (5.7) a general state equation and (5.8) the effect of the unity operator. We now introduce some abbreviations, i.e. we write the functional calculus symbolically in analogy to vector analysis. In vector analysis the reciprocal system is denoted by upper labels whereas the original system has lower labels or vice versa. Then we define

$$B(\eta_1\ldots\eta_k,j):=B_k(j) \tag{5.9}$$

$$T(\xi_1\ldots\xi_n,j):=B^n(j)$$

and write the operators in the following symbolic notation

$$O(j,j')=\sum_{k,i}B_k(j)\,O_i{}^k\,B^i(j') \tag{5.10}$$

with $O_i^k=\int B^k(j)\,O(j,j')\,B_i(j')\,\mathrm{d}j\,\mathrm{d}j'$. (5.11)

The state functionals are written

$$\mathfrak{T}(j)=\sum_n b^n\,B_n(j) \tag{5.12}$$

and (5.1) now reads

$$\int B_m(j)\,B^n(j)\,\mathrm{d}j=\delta_m^n:=g_m^n. \tag{5.13}$$

In this notation we obtain for (5.6) by the use of (5.13)

$$\int O(j,j'')\,P(j'',j')\,\mathrm{d}j''\equiv\sum_{k,m,n}B_k(j)\,O_m^k\,P_n^m\,B^n(j') \tag{5.14}$$

with

$$O_m^k\,P_n^m:=\int O_m^k(\xi_1\ldots\xi_k;\eta_1\ldots\eta_m)\times P_n^m(\eta_1\ldots\eta_m,\zeta_1\ldots\zeta_n)\,\mathrm{d}\eta_1\ldots\mathrm{d}\eta_m \tag{5.15}$$

and for (5.7)

$$\int O(j,j')\,\mathfrak{T}(j')\,\mathrm{d}j'\equiv\sum_{k,m}B_k(j)\,O_m^k\,b^m=\sum_l B_l(j)\,r^l. \tag{5.16}$$

Multiplying (5.16) from the left by $B^n(j)$ and integrating over $j$ we obtain

$$\sum_m O_m^n\,b^m=r^n\quad(n=1,\ldots,\infty). \tag{5.17}$$

Because we used initially differential operators it is interesting to show their equivalence to integral operators. We assume the following differential representation.

$$O\left(j,\frac{\delta}{\delta j}\right)\mathfrak{T}(j)=\mathfrak{R}(j). \tag{5.18}$$

For the transition to the integral representation we multiply (5.18) by $B^n(j)$, integrate over $j$, and substitute the unity operator (5.4) symbolically written

$$\delta(j,j')=\sum_n B_n(j)\,B^n(j') \tag{5.19}$$

according to the rules (5.16). Then we obtain for the lefthand side of (5.18)

$$\int B^n(j)\,O\left(j,\frac{\delta}{\delta j}\right)\delta(j,j')\,\mathfrak{T}(j')\,\mathrm{d}j\,\mathrm{d}j' \tag{5.20}$$

$$=\sum_m\int B^n(j)\,O\left(j,\frac{\delta}{\delta j}\right)B_m(j)\,\mathrm{d}j\int B^m(j')\,\mathfrak{T}(j')\,\mathrm{d}j'.$$

Remembering that the scalar product of $B^m$ with $\mathfrak{T}$ and $B^n$ with $\mathfrak{R}$ just gives $b^m$ respectively $r^n$ (5.18) goes over in the appropriate integral representation (5.17) if we put

$$O(j,j'):=O\left(j,\frac{\delta}{\delta j}\right)\delta(j,j') \tag{5.21}$$

be definition. Using this definition for the calculation of the integral representation (5.2) and (5.3) of a differential operator $O(j, \delta/\delta j)$ the equivalence of both representations is secured.

The advantage of the integral representation obviously stems from the fact that it is invariant against the choice of special representations and that one may apply due to the analogy with vector analysis the apparatus of vector analysis in functional space if necessary. For example we may define

$$\int B_m(j)\, B_n(j)\, \mathrm{d}j = : g_{mn} \tag{5.22}$$

as metrical tensor of the base functionals $B_m(j)$ in functional space for raising and lowering indices like

$$O_{mi} = \int B_m(j)\, O(j, j')\, B_i(j')\, \mathrm{d}j\, \mathrm{d}j' = \sum_k g_{mk}\, O_i^k . \tag{5.23}$$

This possibility is of special importance because the properties of functional operators can be obscured by an unfortunate choice of the base functionals. An example is provided by any selfadjoint operator $A$ whose integral representation $A_k{}^i$ is not symmetrical as long as the reciprocal set is not identical with the original set, i.e. as long as we do not use an orthonormalized set. On the other hand $A_{ki}$ is symmetric even if the basic set is not orthomalized.

## 6. The Dyson Representation

As a first step towards the solution of the functional Eq. (3.5) we expand $\mathfrak{T}_\varrho(j)$ into a series of Dyson functionals. As these base functionals are normalizable at least in the one-time limit they cannot be completely wrong. Moreover this expansion is of special interest as all calculations in nonlinear spinor theory have been done in this representation. When we write the functional $\mathfrak{T}_\varrho(j)$ in the symbolic notation[13]

$$\mathfrak{T}_\varrho(j) = \sum_{k=1}^{\infty} \varphi^k(\varrho)\, D_k(j) \tag{6.1}$$

and denote the reciprocal system (4.7) by $D^l(j)$ the operators of Eq. (3.5) are expanded in the $D_k(j)$ and $D^l(j)$ functionals, giving

$$A_i^k : = \int D^k(j)\, j(t) \frac{\delta}{\delta j(t)}\, D_i(j)\, \mathrm{d}t\, \mathrm{d}j \tag{6.2}$$

and

$$C_i^k : = \int D^k(j)\, j(t)\, G(t-t') \times N\left(j(t'), \frac{\delta}{\delta j(t')}\right) D_i(j)\, \mathrm{d}t\, \mathrm{d}t'\, \mathrm{d}j . \tag{6.3}$$

In this representation Eq. (3.5) then reads

$$\sum_{l=1}^{\infty} (A_l^k - C_l^k)\, \varphi^l(\varrho) = 0 \quad (k = 1, \ldots, \infty) . \tag{6.4}$$

Of course, nothing prevents us from using the operators $A_{kl}$ and $C_{kl}$ which do not lose their symmetry properties in the chosen representation. But we treat (6.4) specially because (6.4) corresponds exactly to the usual field theoretic representation. After some lengthy calculations given in Appendix II which only use those properties of the $D_k$ and $D^l$ sets being mentioned in Section 4 we obtain the explicit expressions

$$\begin{aligned}
\sum_l (A_l^m - C_l^m)\, \varphi^l(\varrho) \equiv\; & m\, \varphi_m(t_1 \ldots t_m) + \sum_{\lambda_1=1}^{m} \int G(t_{\lambda_1} - \alpha)\, \varphi_{m+2}(t_{\lambda_2} \ldots t_{\lambda m}\, \alpha\, \alpha\, \alpha)\, \mathrm{d}\alpha \\
& + 3 \sum_{(\lambda_1, \lambda_2)=1}^{m} \int K(t_{\lambda_1} t_{\lambda_2} \alpha)\, \varphi_m(t_{\lambda_3} \ldots t_{\lambda_m}\, \alpha\, \alpha)\, \mathrm{d}\alpha + \sum_{(\lambda_1, \lambda_2)=1}^{m} h(t_{\lambda_1} t_{\lambda_2})\, \varphi_{m-2}(t_{\lambda_3} \ldots t_{\lambda_m}) \\
& + 3 \sum_{(\lambda_1 \lambda_2 \lambda_3)=1}^{m} \int H_1(t_{\lambda_1} t_{\lambda_2} t_{\lambda_3} \alpha)\, \varphi_{m-2}(t_{\lambda_4} \ldots t_{\lambda_m}\, \alpha)\, \mathrm{d}\alpha + \sum_{(\lambda_1 \lambda_2 \lambda_3 \lambda_4)=1}^{m} H_2(t_{\lambda_1} t_{\lambda_2} t_{\lambda_3} t_{\lambda_4})\, \varphi_{m-4}(t_{\lambda_5} \ldots t_{\lambda_m}) = 0
\end{aligned} \tag{6.5}$$

with the definitions

$$\begin{aligned}
h(t_1 t_2) : &= \operatorname*{sym}_{t_1 t_2} \left[ F(t_1 - t_2) + \frac{1}{i}\, G(t_1 - t_2) \right], \\
K(t_1 t_2 \alpha) : &= \operatorname*{sym}_{t_1 t_2} G(t_1 - \alpha)\, F(t_2 - \alpha), \\
H_1(t_1 t_2 t_3 \alpha) : &= \operatorname*{sym}_{t_1 t_2 t_3} G(t_1 - \alpha)\, F(t_2 - \alpha)\, F(t_3 - \alpha), \\
H_2(t_1 t_2 t_3 t_4) : &= \operatorname*{sym}_{t_1 t_2 t_3 t_4} \int G(t_1 - \alpha)\, F(t_2 - \alpha)\, F(t_3 - \alpha)\, F(t_4 - \alpha)\, \mathrm{d}\alpha
\end{aligned} \tag{6.6}$$

[13] Usually in quantum field theory the Dyson expansion is written as a functional transformation $\mathfrak{T}_\varrho(j) = \Phi_\varrho(j) \exp[-\frac{1}{2} j \cdot F \cdot j]$ from the $\mathfrak{T}_\varrho(j)$-functional to the $\Phi_\varrho(j)$-functional. But this description is dangerous, because by a clumsy application of the transformation the functional operators can loose their selfadjount property.

where ,,sym" means symmetrization in all indices and $(\lambda_1 \ldots \lambda_k)$ means the sum over all possible combinations of $k$ elements out of $m$ elements independently of their sequence. The functions $G$ and $F$ are defined by (3.3) and in Sect. 4. As all FEYNMAN integrals are to be calculated in momentum space it is still convenient to perform a FOURIER transformation on (6.5). Denoting all FOURIER transforms with a tilde, we obtain the FOURIER transformed Eqs. (6.7) and (6.8).

$$\begin{aligned} m\,\tilde{\varphi}_m(q_1\ldots q_m) &+ \frac{1}{(2\pi)^2}\sum_{\lambda_1=1}^{m}\tilde{G}(q_{\lambda_1})\int\tilde{\varphi}_{m+2}(q_{\lambda_2}\ldots q_{\lambda_m};\,\xi\,\eta\,\zeta)\,\delta(\xi+\eta+\zeta-q_{\lambda_1})\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta \\ &+ 3\,\frac{1}{(2\pi)^2}\sum_{(\lambda_1,\lambda_2)=1}^{m}\int\tilde{K}(q_{\lambda_1}q_{\lambda_2}\xi)\,\tilde{\varphi}_m(q_{\lambda_3}\ldots q_{\lambda_m}\eta\,\zeta)\,\delta(\xi+\eta+\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta \\ &+ \sum_{(\lambda_1,\lambda_2)=1}^{m}\tilde{h}(q_{\lambda_1}q_{\lambda_2})\,\tilde{\varphi}_{m-2}(q_{\lambda_3}\ldots q_{\lambda_m}) \\ &+ 3\,\frac{1}{2\pi}\sum_{(\lambda_1\lambda_2\lambda_3)=1}^{m}\int\tilde{H}_1(q_{\lambda_1}q_{\lambda_2}q_{\lambda_3}\xi)\,\tilde{\varphi}_{m-2}(q_{\lambda_4}\ldots q_{\lambda_m}\eta)\,\delta(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &+ \sum_{(\lambda_1\lambda_2\lambda_3\lambda_4)=1}^{m}\tilde{H}_2(q_{\lambda_1}q_{\lambda_2}q_{\lambda_3}q_{\lambda_4})\,\tilde{\varphi}_{m-4}(q_{\lambda_5}\ldots q_{\lambda_m}) = 0 \end{aligned} \tag{6.7}$$

with

$$\begin{aligned} \tilde{h}(q_1q_2) &:= 4\pi\left[\tilde{F}(q_1)+\frac{1}{i}\tilde{G}(q_1)\right]\delta(q_1+q_2) := -\,h(q_1)\,\delta(q_1+q_2)\,, \\ \tilde{K}(q_1q_2\xi) &:= \operatorname*{sym}_{q_1q_2}\tilde{G}(q_1)\,\tilde{F}(q_2)\,2\pi\,\delta(q_1+q_2+\xi) := -\,(2\pi)^2 k(q_1q_2)\,\delta(q_1+q_2+\xi)\,, \\ \tilde{H}_1(q_1q_2q_3\xi) &:= \operatorname*{sym}_{q_1q_2q_3}\tilde{G}(q_1)\,\tilde{F}(q_2)\,\tilde{F}(q_3)\,2\pi\,\delta(q_1+q_2+q_3+\xi) \\ &:= -\,2\pi\,h_1(q_1q_2q_3)\,\delta(q_1+q_2+q_3+\xi)\,, \\ \tilde{H}_2(q_1q_2q_3q_4) &:= \operatorname*{sym}_{q_1q_2q_3q_4}\tilde{G}(q_1)\,\tilde{F}(q_2)\,\tilde{F}(q_3)\,\tilde{F}(q_4)\,2\pi\,\delta(q_1+q_2+q_3+q_4) \\ &:= -\,h_2(q_1q_2q_3q_4)\,, \quad g(q) := -\,\frac{1}{(2\pi)^2}\,\tilde{G}(q) \end{aligned} \tag{6.8}$$

where $\tilde{G}$ and $\tilde{F}$ are FOURIER transforms of $G$ and $F$ which are discussed in Sect. 10. We still have to observe the subsidiary condition (2.8) for stationary functionals. We do not transform it explicitly because we are going to show in Sect. 8 that by means of our solution procedure this condition is satisfied automatically.

## 7. General Solution Procedure

The system (6.7) is a system of integral equations but not in the common sense. Irrespective of the fact that it is an infinite system (6.7) contains a lot of $\delta$-functions and of "unbounded" variables, i.e. variables over which no integration has been carried out. Therefore we have first to prepare our system before we are able to integrate it. The first step consists in removing all $\delta$-functions from (6.7) which are integrated with the unknown $\varphi$-functions. Defining the functions

$$\varphi_m^1(q_1\,|\,q_2\ldots q_{m-1}) := \int\tilde{\varphi}_m(\xi,\,q_1-\xi,\,q_2\ldots q_{m-1})\,\mathrm{d}\xi \tag{7.1}$$

we may write the system (6.7) in the following way

$$\begin{aligned} m\,\tilde{\varphi}_m(q_1\ldots q_m) &= \sum_{\lambda_1=1}^{m} g(q_{\lambda_1})\int\varphi^1{}_{m+2}(q_{\lambda_1}-\eta\,|\,\eta,\,q_{\lambda_2}\ldots q_{\lambda_m})\,\mathrm{d}\eta \\ &\quad + 3\sum_{(\lambda_1,\lambda_2)=1}^{m} k(q_{\lambda_1}q_{\lambda_2})\,\varphi^1{}_m(q_{\lambda_1}+q_{\lambda_2}\,|\,q_{\lambda_3}\ldots q_{\lambda_m}) + r_m(q_1\ldots q_m) \end{aligned} \tag{7.2}$$

with

$$
\begin{aligned}
r_m(q_1 \dots q_m) := & \sum_{(\lambda_1, \lambda_2)=1}^{m} h(q_{\lambda_1})\, \delta(q_{\lambda_1} + q_{\lambda_2})\, \tilde{\varphi}_{m-2}(q_{\lambda_3} \dots q_{\lambda_m}) \\
& + 3 \sum_{(\lambda_1 \lambda_2 \lambda_3)=1}^{m} h_1(q_{\lambda_1} q_{\lambda_2} q_{\lambda_3})\, \tilde{\varphi}_{m-2}(q_{\lambda_1} + q_{\lambda_2} + q_{\lambda_3}, q_{\lambda_4} \dots q_{\lambda_m}) \\
& + \sum_{(\lambda_1 \lambda_2 \lambda_3 \lambda_4)=1}^{m} h_2(q_{\lambda_1} q_{\lambda_2} q_{\lambda_3} q_{\lambda_4})\, \tilde{\varphi}_{m-4}(q_{\lambda_5} \dots q_{\lambda_m}) \, .
\end{aligned}
\tag{7.3}
$$

From (7.2) follows that one has to pay for the removal of the $\delta$-functions by the introduction of the new unknown functions $\varphi_m^1$. Of course, these functions are calculable. Applying the "contraction" operation (7.1) to the system (7.2) one obtains a system for the $\varphi_m^1$-functions but this system now contains double contracted $\varphi$-functions etc. The necessary procedure for obtaining a closed system of integral equations can be presented in a systematic way. To perform this we specialize to the case of state functionals for stationary states of even parity. Then only even indices $m = 2, 4 \dots$ do occur in (6.7) or (7.2) respectively. The case of odd parity can be treated in complete analogy. Assuming now a completely symmetric set of FOURIER transforms $\tilde{f}_{2m}(q_1, \dots, q_{2m})$ of a state functional $\mathfrak{E}(j)$ we can define the following contraction functions

$$
\begin{aligned}
f_{2m}^k(q_1 \dots q_k \,|\, q_{k+1} \dots q_{2m-k}) & := P_k \dots P_1 \tilde{f}_{2m}(q_1 \dots q_{2m}) \\
& := \int \tilde{f}_{2m}(\xi_1 \dots \xi_k, q_1 - \xi_1 \dots q_k - \xi_k, q_{k+1} \dots q_{2m-k})\, \mathrm{d}\xi_1 \dots \mathrm{d}\xi_k
\end{aligned}
\tag{7.4}
$$

with $\tilde{f}_{2m} := f_{2m}^0$. Now we apply the contraction operators $P_k \dots P_1$ $(k = 1 \dots m)$ to the Eqs. (7.2) for even $m$. This results in the system of equations

$$
\begin{aligned}
& [m - 3 \sum_{j=1}^{k} k^1(q_j)]\, \varphi_{2m}^k(q_1 \dots q_k \,|\, q_{k+1} \dots q_{2m-k}) \\
& = \sum_{\lambda_1=k+1}^{2m-k} g(q_{\lambda_1}) \int \varphi_{2m+2}^{k+1}(q_1 \dots q_k, q_{\lambda_1} - \eta \,|\, \eta\, q_{\lambda_2} \dots q_{\lambda_{2(m-k)}})\, \mathrm{d}\eta \\
& + 2 \sum_{\mu_1=1}^{k} \int g(q_{\mu_1} - \xi)\, \varphi_{2m+2}^k(q_{\mu_2} \dots q_{\mu_k}, q_{\mu_1} - \eta - \xi \,|\, \xi, \eta, q_{k+1} \dots q_{2m-k})\, \mathrm{d}\xi\, \mathrm{d}\eta \\
& + \sum_{(\lambda_1 \lambda_2)=k+1}^{2m-k} 3\, k(q_{\lambda_1} q_{\lambda_2})\, \varphi_{2m}^{k+1}(q_1 \dots q_k; q_{\lambda_1} + q_{\lambda_2} \,|\, q_{\lambda_3} \dots q_{\lambda_{2(m-k)}}) \\
& + 2 \sum_{\lambda_1=k+1}^{2m-k} \sum_{\mu_1=1}^{k} 3 \int k(q_{\lambda_1}, q_{\mu_1} - \xi)\, \varphi_{2m}^k(q_{\mu_2} \dots q_{\mu_k}, q_{\lambda_1} + q_{\mu_1} - \xi \,|\, \xi, q_{\lambda_2} \dots q_{\lambda_{2(m-k)}})\, \mathrm{d}\xi \\
& + 4 \sum_{(\mu_1 \mu_2)=1}^{k} 3 \int k(q_{\mu_1} - \xi, q_{\mu_2} - \eta)\, \varphi_{2m}^{k-1}(q_{\mu_3} \dots q_{\mu_k}, q_{\mu_1} + q_{\mu_2} - \xi - \eta \,|\, \xi\, \eta\, q_{k+1} \dots q_{2m-k})\, \mathrm{d}\xi\, \mathrm{d}\eta \\
& + r_{2m}^k(q_1 \dots q_k \,|\, q_{k+1} \dots q_{2m-k}) \qquad (m = 1, \dots, \infty;\ k = 1, \dots, m) \, .
\end{aligned}
\tag{7.5}
$$

The details of its derivation are given in Appendix III. It is remarkable that the system (7.5) is a closed system in the unknown functions $\varphi_{2m}^k$ $(m = 1, \dots, \infty;\ k = 0, \dots, m)$ because the equations (7.5) become for $k = m$

$$
\begin{aligned}
& [m - \sum_{j=1}^{m} k^1(q_j)]\, \varphi_{2m}^m(q_1 \dots q_m \,|\,) \\
& = 2 \sum_{\mu_1=1}^{m} \int g(q_{\mu_1} - \xi)\, \varphi_{2m+2}^m(q_{\mu_2} \dots q_{\mu m};\ q_{\mu_1} - \xi - \eta \,|\, \xi\, \eta)\, \mathrm{d}\xi\, d\eta \\
& + 4 \sum_{(\mu_1 \mu_2)=1}^{m} \int k(q_{\mu_1} - \xi, q_{\mu_2} - \eta)\, \varphi_{2m}^{m-1}(q_{\mu_3} \dots q_{\mu_m};\ q_{\mu_1} + q_{\mu_2} - \xi - \eta \,|\, \xi \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta \\
& + r_{2m}^m(q_l \dots q_m \,|\,) \, .
\end{aligned}
\tag{7.6a}
$$

The last term $r^m_{2m}(q_1 \ldots q_m|)$ reads in a symbolic notation, the precise details of which being given in the Appendix III,

$$r^m_{2m}(q_1 \ldots q_m|) = \sum_{\mu=m-1}^{m-2} \sum_{\nu=\mu}^{\mu+2} W^{m,\nu}_{m\mu} \varphi^\nu_{2\mu}. \tag{7.6 b}$$

Therefore the system for the contraction functions $\varphi^k_{2m}$ $(k = 0, \ldots, m)$ terminates with $\varphi^m_{2m}$ and no higher contraction term occurs. So (7.5) can be used instead of (6.7).

For the solution of the infinite systems we now use approximate functionals

$$\mathfrak{T}^N_\varrho(j) := \sum_{m=1}^{N} \varphi_{2m}(N) D_{2m}(j). \tag{7.7}$$

We do not discuss here the question of convergence for $N$ going to infinity, but only the calculation of (7.7) for arbitrary $N$. Formally the use of (7.7) can be defined by putting $\varphi_{2\alpha} = 0$ for $\alpha > N$ and then calculating $\varphi_2 \ldots \varphi_{2N}$ from the first $N$ equations of (6.7). By our contraction procedure we transformed (6.7) into (7.5) and perform calculations with this system. From (7.4) follows in this case that $\varphi^k_{2\alpha} = 0$ for $\alpha > N$ and $k = 0 \ldots \alpha$. Therefore the truncation procedure for (7.5) is defined by putting $\varphi^k_{2\alpha} = 0$ for $\alpha > N$, $k = 0 \ldots \alpha$ and then calculating $\varphi^k_{2\alpha}$ for

$$\alpha = 1, \ldots, N \quad \text{and} \quad k = 0, \ldots, \alpha$$

from the corresponding equations of (7.5).

Writing the Eqs. (7.5) in symbolic form we have the system

$$\varphi^k_{2m} = \sum_{n=m-2}^{m+1} \sum_{l=k-4}^{k+1} W^{kl}_{mn} \varphi^l_{2n} \begin{pmatrix} m = 1, \ldots, \infty \\ k = 0, \ldots, m \end{pmatrix} \tag{7.8}$$

and obtain for the calculation of (7.7) the truncated system

$$\varphi^k_{2m}(N) = \sum_{\substack{n=m-2 \\ \leq N}}^{m+1} \sum_{l=k-4}^{k+1} W^{kl}_{mn} \varphi^l_{2k}(N) \begin{pmatrix} m = 1, \ldots, N \\ k = 0, \ldots, m \end{pmatrix}. \tag{7.9}$$

Then we have to integrate (7.9) explicitly. This will be done in details in the following Sect. 9 for the case of $N = 1$ and $N = 2$. Here we only want to sketch the general method. To do this it is not necessary to write down all indices explicitly. We rather use a shorthand notation. We define a subset of functions $\varphi^k_{2m}$ by the symbol

$$\boldsymbol{\Phi}(\alpha, \beta) := \varphi^k_{2m} \begin{pmatrix} m = \alpha, \ldots, \beta \\ k = 0, 1, \ldots, m \end{pmatrix} \tag{7.10}$$

and for any operator $O^{kl}_{mn}$ we define its projection on this subset by

$$\boldsymbol{O}(\alpha, \beta) := O^{kl}_{mn} \begin{pmatrix} m, n = \alpha, \ldots, \beta \\ k, l = 0, 1, \ldots, m, n \end{pmatrix}. \tag{7.11}$$

Now we start with the lowest possible equation of (7.9) for $\varphi^0_2$, which reads

$$\varphi^0_2(N) = W^{00}_{11} \varphi^0_2(N) + W^{01}_{11} \varphi^1_2(N) + W^{01}_{12} \varphi^1_4(N) \tag{7.12}$$

where all other terms drop out according to the structure of (7.5). Then the remaining equations of (7.9) read in the notation of (7.10) and (7.11)

$$\boldsymbol{\Phi}_N(2, N) = \boldsymbol{W}(2, N) \cdot \boldsymbol{\Phi}_N(2, N) + \mathfrak{C} \tag{7.13}$$

where $\mathfrak{C}$ contains all terms with $\varphi^0_2$ and $\varphi^1_2$ i.e.

$$\mathfrak{C} := \sum_{\varrho=0}^{1} W^{k\varrho}_{m1} \varphi^\varrho_2(N) \begin{pmatrix} m = 2, \ldots, N \\ k = 0, \ldots, m \end{pmatrix}. \tag{7.14}$$

Then we construct the Green function for (7.13) namely

$$\boldsymbol{G}(2, N) := [\boldsymbol{1}(2, N) - \boldsymbol{W}(2, N)]^{-1} \tag{7.15}$$

and apply it to (7.13) obtaining so

$$\boldsymbol{\Phi}_N(2, N) = \boldsymbol{G}(2, N) \mathfrak{C}. \tag{7.16}$$

From all these functions we only need $\varphi_4{}^1$ which reads according to (7.16)

$$\varphi^1_4(N) = \sum_{nl} G^{1l}_{2n}(N) \sum_{\varrho=0}^{1} W^{l\varrho}_{n1} \varphi^\varrho_2(N) := S_0 \varphi^0_2 + S_1 \varphi^1_2. \tag{7.17}$$

This inserted into (7.12) results in

$$\varphi^0_2(N) = [W^{00}_{11} + W^{01}_{12} S_0] \varphi^0_2(N) + [W^{01}_{11} + W^{01}_{12} S_1] \varphi^1_2 \tag{7.18}$$

or by inversion

$$\varphi^0_2(N) = \frac{[W^{01}_{11} + W^{01}_{12} S_1] \varphi^1_2(N)}{1 - W^{00}_{11} - W^{01}_{12} S_0}. \tag{7.19}$$

In abbreviated form this can be written

$$\varphi^0_2(N) = Q(N) \varphi^1_2(N). \tag{7.20}$$

When we apply the contraction operation on (7.20) we have

$$[1 - Q^1(N)] \varphi^1_2(N) = 0 \tag{7.21}$$

and after introduction of center of gravity coordinates by

$$\varphi^1_2(q) = \int \varphi^0_2(q - \xi, \xi) \, \mathrm{d}\xi = \text{const} \cdot \delta(q - \omega) \tag{7.22}$$

we obtain an eigenvalue equation for the calculation of the approximate eigenvalue $\omega_N$ corresponding to the approximated functional (7.7)

$$Q^1(N;\,\omega_N) = 1\,. \tag{7.22}$$

As one easily recognizes, the main calculational problem is the construction of $G(2, N)$. We want to demonstrate how to do this.

We start with solving the truncated system $\varphi_{2m}^k$ $(k = 0, \ldots, m)$ for the highest fixed index $m = N$ and then for the highest possible $k = m = N$.

We receive from Eq. (7.6a)

$$\varphi_{2N}^N = W_{NN}^{NN-1}\,\varphi_{2N}^{N-1} + r_{2N}^N[\varphi_{2(N-1)}^k, \varphi_{2(N-2)}^k]\,. \tag{7.24}$$

Inserting this in the next lower equation of (7.9) having $k = N-1$ we get an equation for $\varphi_{2N}^{N-1}$ having the structure written in an abbreviated form

$$[1 - K_{2N}^{(N-1)}]\,\varphi_{2N}^{N-1} = W\,{}_{N}^{N-1}\,{}_{N}^{N-2} \cdot \varphi_{2N}^{N-2} + R_{2N}^{(N-1)}[\varphi_{2(N-1)}^k, \varphi_{2(N-2)}^k] \tag{7.25}$$

with the solution

$$\varphi_{2N}^{N-1} = \Gamma_{2N}^{(N-1)}\,W_{N}^{N-1}\,{}_{N}^{N-2} \cdot \varphi_{2N}^{N-2} + \Gamma_{2N}^{(N-1)}\,R_{2N}^{(N-1)}[\varphi_{2(N-1)}^k, \varphi_{2(N-2)}^k] \tag{7.26}$$

where $\Gamma_{2N}^{(N-1)}$ is the resolvent of $(1 - K_{2N}^{(N-1)})$. Then $\varphi_{2N}^{N-1}$ can be inserted into the equation of (7.9) for $k = N-2$ resulting in a new resolvent operator $\Gamma_{2N}^{(N-2)}$ and so forth. It can thus be seen that in the case $m = N$ we have to construct $N$ resolvents $\Gamma_{2N}^{(k)}$ $(k = 0, \ldots, N-1)$ to obtain $\varphi_{2N}^0$ as a functional of $\varphi_{2(N-1)}^k$ and $\varphi_{2(N-2)}^k$. From $\varphi_{2N}^0$ we can get the whole $\varphi_{2N}^k$-system $(k = 0, \ldots, N)$ by simple integrations according to its definitions (7.4).

Now we can use the next lower equation system of (7.9) i.e. that for $m = N-1$ and insert the known functions $\varphi_{2N}^k$ $(k = 0, \ldots, N)$ which are given functionals of $\varphi_{2(N-1)}^k$ and $\varphi_{2(N-2)}^k$. Then we obtain in abbreviated form the equation system for $\varphi_{2(N-1)}^k$ $(k = 0, \ldots, N-1)$

$$\varphi_{2(N-1)}^k = \sum_{\nu=k-1}^{k+1} W^{(1)}\,{}_{N-1}^{k}\,{}_{N-1}^{\nu} \cdot \varphi_{2(N-1)}^{\nu} + r_{2(N-1)}^k[\varphi_{2(N-2)}^n, \varphi_{2(N-3)}^n] \tag{7.27}$$

which has the structure of the equation system (7.9) for $N-1$. Thus we can use the same procedure as mentioned above and have to construct $N-1$ resolvents to get the functions $\varphi_{2(N-1)}^k$ which can be inserted into the equation system (7.9a) for $m = N-2$ and so forth. Finally we terminate with the desired $\varphi_4^0$, $\varphi_4^1$, $\varphi_4^2$-system after the $\varphi_6^k$ have been inserted. If this system is solved one obtains (7.17) and therefore by the procedure described one has constructed the resolvent operator $G_{2n}^{11}(N)$ which is only of interest. Of course, the remaining $G_{mn}^{k1}(N)$ $(m = 3, \ldots, N,$ $k = 0, \ldots, m)$ can be constructed, if necessary, by the same method. As one recognizes easily the construction of $G_{mn}^{k1}(N)$ requires the construction of $N(N-1)/2$ partial resolvents. In Sect. 9 and 10 this method will be discussed in detail for $N = 1$ and $N = 2$ and the existence problem of the partial resolvents is examined.

## 8. The Condition of Stationarity

Stationary functionals are characterized by the subsidiary condition (2.8). Therefore we have to demonstrate that the solution procedure sketched in the preceding section and leading to an eigenvalue equation does satisfy (2.8) i.e. that the eigenvalues $\omega_N$ calculated according to Sect. 7 are the required ones by the eigenvalue condition (2.8). To do this, we first represent (2.8) by Dyson functionals. Using the general Dyson expansion (6.1) for the state functional, Eq. (2.8) can be written in our symbolic notation

$$\sum_{l=1}^{\infty} P_l^k\,\varphi^l(\varrho) = -\,i\,\omega_\varrho\,\varphi^k(\varrho) \tag{8.1}$$

with

$$P_i^k := \int D^k(j)\,j(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\delta}{\delta j(t)}\,\mathrm{d}t\,D_i(j)\,\mathrm{d}j\,. \tag{8.2}$$

Explicit evaluation gives

$$P_i^k = \delta_i^k\,\delta(t_1 - t_1') \ldots \delta(t_k - t_k')\sum_{r=1}^{k}\frac{\mathrm{d}}{\mathrm{d}t_k} \tag{8.3}$$

i.e. $P_j^k$ is a diagonal operator. Transformation of (8.1) in Fourier space then gives the equation

$$[\sum_{i=1}^{n} q_i]\,\tilde{\varphi}_n(q_1 \ldots q_n) = \omega_\varrho\,\tilde{\varphi}_n(q_1 \ldots q_n)\,. \tag{8.4}$$

Specializing on states of even parity, we may apply the contraction operation of Sect. 7 on (8.4) and obtain in this way the subsidiary conditions

$$[\sum_{i=1}^{2m-k} q_i]\,\varphi_{2m}^k(q_1 \ldots q_k \,|\, q_{k+1} \ldots q_{2m-k}) = \omega_\varrho\,\varphi_{2m}^k(q_1 \ldots q_k \,|\, q_{k+1} \ldots q_{2m-k}) \tag{8.5}$$

$$(m = 1, \ldots, \infty;\; k = 0, \ldots, m)\,.$$

So far Eqs. (8.1) to (8.5) are valid for the exact stationary state functionals. Assuming now an approximate functional (7.7), not only the dynamical equations (7.8) have to be truncated but also the Eq. (8.1) respectively (8.5).

This gives

$$\sum_{l=1}^{N} P_l^k \varphi^l(N) = \sum_{l=1}^{\infty} P_e^k \varphi^l(N) - i\,\omega_\varrho \varphi^k(N) \qquad (k=1,\ldots,N) \quad (8.6)$$

due to the diagonal structure of $P_l^k$. Therefore (2.8) has to be exactly valid for the approximate functionals too. Of course, then the equations (8.5) have to be satisfied by the $\varphi_{2m}^k(N)$ also. Now the problem can be formulated as follows: According to Sect. 7 only $\varphi_2^1$ and $\varphi_2^0$ are properly chosen to satisfy (8.5). If one calculates the higher $\varphi$-functions by the outlined procedure do then all $\varphi_{2m}^k(N)$ satisfy (8.5) automatically or not? To make clear that this condition is satisfied in any approximation step we use the symbolic notation of Sect. 7. Defining the total sets $\boldsymbol{\Phi}(1,\infty) := \boldsymbol{\Phi}$ and the corresponding operator $\boldsymbol{O}(1,\infty) := \boldsymbol{O}$ the subsidiary condition (8.5) may be written

$$\boldsymbol{P}_0 \boldsymbol{\Phi}_\varrho = \omega_\varrho \boldsymbol{\Phi}_\varrho \quad (8.7)$$

for the exact physical state functionals. Therefore the operator of Eq. (7.8) written symbolically

$$\boldsymbol{\Phi}_\varrho = \boldsymbol{W} \boldsymbol{\Phi}_\varrho \quad (8.8)$$

has to commute with $\boldsymbol{P}_0$ and it can easily be seen that this condition is fullfilled.

$$[\boldsymbol{P}_0, \boldsymbol{W}]_- = 0\,. \quad (8.9)$$

Observing now, that $\boldsymbol{P}_0$ is a diagonal operator, the commutation relation (8.9) has to be valid also for any subset indices $\alpha$, $\beta$ namely

$$[\boldsymbol{P}_0(\alpha,\beta), \boldsymbol{W}(\alpha,\beta)]_- = 0\,. \quad (8.10)$$

Then from (8.10) follows also

$$[\boldsymbol{P}_0(2,N), [\boldsymbol{1}(2,N) - \boldsymbol{W}(2,N)]]_- = 0 \quad (8.11)$$

and from this one concludes easily, that

$$[\boldsymbol{P}_0(2,N), \boldsymbol{G}(2,N)]_- = 0 \quad (8.12)$$

has to be valid too. Assuming now the inhomogenious term $\mathfrak{G}$ of (7.13) to be an eigenfunctional of $\boldsymbol{P}_0(2,N)$, namely

$$\boldsymbol{P}_0(2,N)\,\mathfrak{G} = \omega_N \mathfrak{G} \quad (8.13)$$

we then have from (7.16)

$$\boldsymbol{P}_0(2,N)\,\boldsymbol{\Phi}_N(2,N) = \boldsymbol{G}(2,N)\,\boldsymbol{P}_0(2,N)\,\mathfrak{G} = \omega_N \boldsymbol{G}(2,N)\,\mathfrak{G} = \omega_N \boldsymbol{\Phi}_N(2,N)\,. \quad (8.14)$$

Therefore $\boldsymbol{\Phi}_N(2,N)$ satisfies the required subsidiary condition if (8.13) is fulfilled. Now our general solution method requires $\varphi_2^0$ and $\varphi_2^1$ to be eigenfunctions of $\boldsymbol{P}_0(1,1)$ according to (7.22). Inserting (7.22) into (7.14) one easily proves (8.13). Therefore by our solution procedure any approximate functional automatically satisfies (2.8).

There is still another point of view, expressing the same fact. According to the subsidiary condition (8.5) we should have solutions like

$$\varphi_{2m}^k(q_1\ldots q_k\,|\,q_{k+1}\ldots q_{2m-k}) = \delta\Big(\sum_{i=1}^{2m-k} q_i - \omega_\varrho\Big)\,\chi_{2m}^k(q_1\ldots q_k\,|\,q_{k+1}\ldots q_{2m-k})\,. \quad (8.15)$$

Roughly this can be written

$$\boldsymbol{\Phi}_\varrho = \delta(\boldsymbol{P}_0 - \omega_\varrho\cdot\boldsymbol{1})\,\chi_\varrho\,. \quad (8.16)$$

If we insert the solution (8.16) of the subsidiary condition (8.7) in the general system (8.8), we have

$$\delta(\boldsymbol{P}_0 - \omega_\varrho\cdot\boldsymbol{1})\,\chi_\varrho = \boldsymbol{W}\,\delta(\boldsymbol{P}_0 - \omega_\varrho\cdot\boldsymbol{1})\,\chi_\varrho = \delta(\boldsymbol{P}_0 - \omega_\varrho\cdot\boldsymbol{1})\,\boldsymbol{W}\chi_\varrho \quad (8.17)$$

according to the commutation relation (8.9). We therefore have to solve the system

$$\chi_\varrho = \boldsymbol{W}\cdot\chi_\varrho \quad (8.18)$$

as long as it expresses a functional relation between the different $\chi_{2m}^k$ and does not influence the range of the variable $q_1,\ldots,q_{2m}$. Especially because the operator $\boldsymbol{W}$ has not been changed, the system (8.18) is formally the same as that which we have got in the preceding section. Using the solution procedure described there we get finally an equation similar to that we have got in (7.21)

$$[1 - \tilde{Q}(N;q)]\,\delta(q-\omega_\varrho) = 0 \quad (8.19)$$

where we have recognized that $\chi_2^1 = \text{const}$ according to (7.22). This results after $q$-integration in Eq. (7.23)

$$\tilde{Q}(N;\omega_\varrho) = Q^1(N;\omega_\varrho) = 1\,. \quad (8.20)$$

Thus, by imposing the translational condition (7.22) on $\varphi_2^1$, transforming it into centre of gravity coordinates, it is guaranteed that the higher $\tilde{\varphi}_{2m}(q_1\ldots q_{2m})$ and its contracted forms are translationally invariant, i.e. they show the structure (8.16).

## 9. Integration of the Approximate Systems

We now explicitly demonstrate the method of integration for the truncated systems (7.7) in the simplest cases, i.e. $N=1$ and $N=2$. For $N=1$ we have $\varphi_2^0$, $\varphi_2^1$ and all other $\varphi_{2m}^k$ are equal to zero. For reasons of simplicitly we denote $\varphi_2^0$ only by $\varphi_2$ and $\varphi_2^1$ by $\overline{\varphi}_2$. Then, according to (7.5), the two equations for $\varphi_2$ and $\overline{\varphi}_2$ read

$$2\,\varphi_2(q_1 q_2) = 3\,k(q_1 q_2)\,\overline{\varphi}_2(q_1+q_2)\,, \tag{9.1}$$

$$2\,\overline{\varphi}_2(q_1) \quad = 3\,\bar{k}(q_1)\,\overline{\varphi}_2(q_1)\,. \tag{9.2}$$

As long as one does not intend to perform norm calculations Eq. (9.1) is not required at all. For the eigenvalue calculation only Eq. (9.2) remains. Putting

$$\overline{\varphi}_2(q_1) = c_0\cdot\delta(q_1-\omega) \tag{9.3}$$

we obtain from (9.2) by inserting (9.3) the secular equation

$$\tfrac{3}{2}\,\bar{k}(\omega) = 1\,. \tag{9.4}$$

This equation corresponds to the lowest N.T.D. equation in nonlinear spinor theory of elementary particles which only contains a local graph. For the numerical evaluation of (9.4) we observe the definition of $k(q_1,q_2)$ given in (6.8). The functions $\tilde{G}$ and $\tilde{F}$ are discussed in Appendix V. Then we obtain for the contracted $k$ of (9.4) by substituting $\tilde{G}$ and an approximated $\tilde{F}_{app}$ from Appendix V

$$\tilde{K}(\omega) = \frac{1}{2\pi}\int \tilde{G}(\omega-\xi)\,\tilde{F}_{app}(\xi)\,\mathrm{d}\xi\cdot 2 = \frac{(a+\omega_1)}{a\cdot\omega_1}\cdot\frac{1}{[\omega^2-(a+\omega_1)^2]}\,. \tag{9.4a}$$

The values of $a$ and $\omega_1$ are also given in Appendix V. Substituting (9.4a) in (9.4) we get the solution

$$\omega_{20}^1 = \sqrt{6}\cdot\omega_1 = 2.8009 \tag{9.4b}$$

which in comparison with the exact value $\omega_{20} = 2.538068$ gives a derivation of roughly 10%.

Now we turn to $N=2$. For $N=2$ the functions $\varphi_2^0$, $\varphi_2^1$, $\varphi_4^0$, $\varphi_4^1$, $\varphi_4^2$ are unequal to zero and all other $\varphi_{2m}^k$ disappear. For simplicity reasons we denote these functions by $\varphi_2$, $\overline{\varphi}_2$, $\varphi_4$, $\overline{\varphi}_4$, $\overline{\overline{\varphi}}_4$. Then, according to (7.5), the system of equations for the calculation of these functions read

$$\begin{aligned}
2\,\varphi_2(q_1 q_2) &= \sum_{\lambda_1=1}^{2} g(q_{\lambda_1})\int\overline{\varphi}_4(q_{\lambda_1}-\eta\,|\,\eta\, q_{\lambda_2})\,\mathrm{d}\eta + 3\,k(q_1 q_2)\,\overline{\varphi}_2(q_1+q_2)\,,\\
[2-3\,\bar{k}(q_1)]\,\overline{\varphi}_2(q_1) &= 2\int g(q_1-\xi)\,\overline{\varphi}_4(q_1-\xi-\eta\,|\,\xi\,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,,\\
4\,\varphi_4(q_1 q_2 q_3 q_4) &= 3\sum_{(\lambda_1,\lambda_2)=1}^{4} k(q_{\lambda_1} q_{\lambda_2})\,\overline{\varphi}_4(q_{\lambda_1}+q_{\lambda_2}\,|\,q_{\lambda_3} q_{\lambda_4}) + r_4(q_1 q_2 q_3 q_4)\,,\\
[4-3\,\bar{k}(q_3)]\,\overline{\varphi}_4(q_3\,|\,q_1 q_2) &= 6\sum_{\lambda_1=1}^{2}\int k(q_{\lambda_1}, q_3-\eta)\,\overline{\varphi}_4(q_{\lambda_1}+q_3-\eta\,|\,q_{\lambda_2}\eta)\,\mathrm{d}\eta\\
&\quad + 3\,k(q_1 q_2)\,\overline{\overline{\varphi}}_4(q_1+q_2, q_3) + \bar{r}_4(q_3\,|\,q_1 q_2)\,,\\
[4-3\,\bar{k}(q_1)-3\,\bar{k}(q_2)]\,\overline{\overline{\varphi}}_4(q_1,q_2) &= 12\int k(q_1-\xi, q_2-\eta)\,\overline{\varphi}_4(q_1+q_2-\xi-\eta\,|\,\xi\,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\
&\quad + \bar{\bar{r}}_4(1,2)
\end{aligned} \tag{9.5}$$

with $g(q) =: (2\pi)^{-2}\,\tilde{G}(q)\,(-1)$. As long as one does not intend to perform norm calculations but eigenvalue calculations not all equations of (9.5) are required. Defining

$$f_1(q_1) := [4-3\,\bar{k}(q_1)]^{-1} \quad\text{and}\quad f_2(q_1,q_2) := [4-3\,\bar{k}(q_1)-3\,\bar{k}(q_2)]^{-1} \tag{9.6}$$

the required equations are the following

$$\begin{aligned}
2\,\varphi_2(q_1,q_2) &= \sum_{\lambda_1=1}^{2} g(q_{\lambda_1})\int\overline{\varphi}_4(q_{\lambda_1}-\eta\,|\,\eta, q_{\lambda_2})\,\mathrm{d}\eta + 3\,k(q_1,q_2)\,\overline{\varphi}_2(q_1+q_2)\,,\\
\overline{\varphi}_4(q_3\,|\,q_1,q_2) &= f_1(q_3)\,\Big\{6\sum_{\lambda_1=1}^{2}\int k(q_{\lambda_1}, q_3-\eta)\,\overline{\varphi}_4(q_{\lambda_1}+q_3-\eta\,|\,q_{\lambda_2}\eta)\,\mathrm{d}\eta\\
&\quad + 3\,k(q_1,q_2)\,\overline{\overline{\varphi}}_4(q_1+q_2, q_3) + \bar{r}_4(q_3\,|\,q_1 q_2)\Big\}\,,\\
\overline{\overline{\varphi}}_4(q_1,q_2) &= f_2(q_1 q_2)\,\Big\{12\int k(q_1-\xi, q_2-\eta)\,\overline{\varphi}_4(q_1+q_2-\xi-\eta\,|\,\xi\,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \bar{\bar{r}}_4(q_1,q_2)\Big\}\,.
\end{aligned} \tag{9.7}$$

Without further manipulation from (9.7) $\bar{\bar{\varphi}}_4$ can be eliminated. Defining the abbreviated expressions

$$g(q_3; q_1 q_2) := 3 f_1(q_3) f_2(q_1 + q_2, q_3) k(q_1 q_2), \tag{9.8}$$
$$R(q_3; q_1 q_2) := f_1(q_3) \{\overline{r_4}(q_3 | q_1 q_2) + 3 k(q_1, q_2) f_2(q_1 + q_2, q_3) \bar{r}_4(q_1 + q_2, q_3)\}$$

we obtain after the elimination of $\bar{\bar{\varphi}}_4$ from (9.7) the system

$$2 \varphi_2(q_1, q_2) = \sum_{\lambda_1=1}^{2} g(q_{\lambda_1}) \int \bar{\varphi}_4(q_{\lambda_1} - \eta | \eta, q_{\lambda_2}) \, d\eta + 3 k(q_1, q_2) \bar{\varphi}_2(q_1 + q_2),$$

$$\bar{\varphi}_4(q_3 | q_1, q_2) = 6 f_1(q_3) \sum_{\lambda_1=1}^{2} \int k(q_{\lambda_1}, q_3 - \eta) \bar{\varphi}_4(q_{\lambda_1} + q_3 - \eta | q_{\lambda_2} \eta) \, d\eta \tag{9.9}$$
$$+ 12 g(q_3; q_1, q_2) \int k(q_1 + q_2 - \xi, q_3 - \eta) \bar{\varphi}_4(q_1 + q_2 + q_3 - \xi - \eta | \xi \eta) \, d\xi \, d\eta$$
$$+ R(q_3; q_1, q_2).$$

Now we should eliminate $\bar{\varphi}_4$ but this cannot be done in one step because in the equation for $\bar{\varphi}_4$ two different types of integrations occur. We first remove the term with the single integrals given by the sum of the right-hand side of (9.9). Using the symbolic notation

$$6 f_1(q_3) \int k(q_{\lambda_1}, q_3 - \eta) \bar{\varphi}_4(q_{\lambda_1} + q_3 - \eta | q_{\lambda_2} \eta) \, d\eta := K_{\lambda_1} \bar{\varphi}_4 \tag{9.10}$$

we write the $\bar{\varphi}_4$ equation in an abbreviated manner

$$\bar{\varphi}_4 = (K_1 + K_2) \bar{\varphi}_4 + f \tag{9.11}$$

where $f$ contains all the other terms occurring in this equation. In the following section the resolvent operator $\Gamma$ for a kernel of the type $K$ in (10.11) is constructed. In our symbolic notation it obeys the resolvent equation

$$\Gamma_{\lambda_1} K_{\lambda_1} = K_{\lambda_1} \Gamma_{\lambda_1} = \Gamma_{\lambda_1} - K_{\lambda_1}. \tag{9.12}$$

For the solution of (9.11) one has to construct a symmetric resolvent operator. According to a proposal of WAHL[14] this operator reads

$$P_1 := [1 + \Gamma_1 + \Gamma_2]. \tag{9.13}$$

Applying $P_1$ to (9.11) we obtain by observing (9.12) the symbolic equation

$$[1 - \Gamma_1 K_2 - \Gamma_2 K_1] \bar{\varphi}_4 = P_1 f. \tag{9.14}$$

Evaluating (9.14) explicitly by means of the definition of the resolvent operator for $K$ in (10.6) we obtain for $\bar{\varphi}_4$ the equation

$$\bar{\varphi}_4(q_3 | q_1 q_2) = \int A(q_3; q_1 q_2, \xi \eta) \bar{\varphi}_4(q_1 + q_2 + q_3 - \xi - \eta | \xi \eta) \, d\xi \, d\eta + F(q_3; q_1 q_2) \tag{9.15}$$

with

$$A(q_3; q_1 q_2, \xi \eta) := 12 g(q_3; q_1 q_2) k(q_1 + q_2 - \xi, \xi - \eta)$$
$$+ 6 \sum_{\lambda_1=1}^{2} \Gamma(q_3; q_{\lambda_2} \xi) f_1(q_{\lambda_2} + q_3 - \xi) k(q_{\lambda_1}, q_{\lambda_2} + q_3 - \xi - \eta) \tag{9.16}$$
$$+ 12 \sum_{\lambda_1=1}^{2} \int \Gamma(q_3; q_{\lambda_2}, q_{\lambda_2} + q_3 - \varrho) g(\varrho; q_{\lambda_1}, q_{\lambda_2} + q_3 - \varrho) k(q_1 + q_2 + q_3 - \xi - \varrho, \varrho - \eta) \, d\varrho$$

and

$$F(q_3; q_1 q_2) := R(q_3; q_1 q_2) + \sum_{\lambda_1=1}^{2} \int \Gamma(q_3; q_{\lambda_1} \varrho) R(q_{\lambda_1} + q_3 - \varrho; q_{\lambda_2} \varrho) \, d\varrho. \tag{9.17}$$

[14] F. WAHL, personal communication.

Now this equation for $\varphi_4$ only contains one type of integration. The solution procedure for such an equation, i.e. the resolvent construction, is also discussed in section 10. According to (10.12) we may write the solution of (9.15)

$$\overline{\varphi}_4(q_3 \mid q_1, q_2) = F(q_3; q_1, q_2) + \int \Lambda(q_3; q_1 q_2, \xi \eta) \, F(q_1 + q_2 + q_3 - \xi - \eta; \xi, \eta) \, \mathrm{d}\xi \, \mathrm{d}\eta \tag{9.18}$$

with $\Lambda$ as the resolvent operator. Now we are able to insert $\overline{\varphi}_4$ into the $\overline{\varphi}_2$-equation of (9.7). By so doing we obtain our last equation

$$2 \varphi_2(q_1 q_2) = 3 k(q_1 q_2) \, \overline{\varphi}_2(q_1 + q_2) + \sum_{\lambda_1=1}^{2} g(q_{\lambda_1}) \, \overline{F}(q_{\lambda_1} \mid q_{\lambda_2}) + \int B(q_1 q_2 \eta \xi) \, F(q_1 + q_2 - \eta - \xi; \eta, \xi) \, \mathrm{d}\eta \, \mathrm{d}\xi \tag{9.19}$$

with

$$B(q_1 q_2 \eta \xi) := \sum_{\lambda_1=1}^{2} g(q_{\lambda_1}) \, \Lambda(q_{\lambda_1} \mid q_{\lambda_2} \eta \xi) \tag{9.20}$$

where the bar across the functions denotes the contraction operation (7.1). This equation is already the desired secular equation in the approximation $N=2$. In order to verify this we only have to evaluate the function $F$ of (9.17) explicitly. By doing so we obtain the following equation

$$\varphi_2(q_1 q_2) = S_1(q_1 q_2) \, \overline{\varphi}_2(q_1 + q_2) + \int S_2(q_1 q_2 \xi) \, \varphi_2(\xi, q_1 + q_2 - \xi) \, \mathrm{d}\xi . \tag{9.21}$$

The functions $S_1$ and $S_2$ are calculable by a rather complicated system of recursion formulas given in Appendix IV. Here we only sketch the further integration of (9.21). According to Sect. 10 there exists a partial resolvent $\Pi$ for $S_2$. Applying it to (9.21) we obtain

$$\varphi_2(q_1 q_2) = [S_1(q_1 q_2) + \int \Pi(q_1 q_2 \xi) \, S_1(\xi, q_1 + q_2 - \xi) \, \mathrm{d}\xi] \, \overline{\varphi}_2(q_1 + q_2) . \tag{9.22}$$

By contraction follows the equation

$$[\overline{S}_1(q_1) + \int \overline{\Pi}(q_1 \mid \xi) \, S_1(\xi, q_1 - \xi) \, \mathrm{d}\xi - 1] \, \overline{\varphi}_2(q_1) = 0 \tag{9.23}$$

and with (9.3) the eigenvalue equation for the approximation $N=2$

$$1 = \overline{S}_1(\omega) + \int \overline{\Pi}(\omega \mid \xi) \, S_1(\xi, \omega - \xi) \, \mathrm{d}\xi . \tag{9.24}$$

It has to be stressed, that (9.24) is an exact equation for the truncated system $N=2$. Further approximations are only necessary for the numerical computation of the partial resolvents $\Gamma$, $\Lambda$ and $\Pi$. As already in the lowest approximations of $\Gamma$, $\Lambda$ and $\Pi$ the numerical work is considerable in solving (9.24), we do not try to give any numerical value here.

Finally we discuss the connection of the integration procedure given here with quantum field theory. Considering this procedure, one immediately recognizes that all steps done in the foregoing are independent of dimension, i.e. any variable $q$ occuring in the preceding equations can be thought as a multidimensional variable $q = (q_1, \ldots, q_n)$ especially as a variable of the four dimensional LORENTZ space. Therefore all operations can immediately be applied to nonlinear spinor theory, provided that all functions appearing are sufficiently regularized. The only difference arises from the fact, that all eigenstates of the anharmonic oscillator are base vectors of the ABELian translational group, whereas in nonlinear spinor theory the rotation group plays a role. Therefore by applying FREDHOLM theory for the construction of the resolvents $\Gamma$, $\Lambda$ and $\Pi$ one probably has to expand first all kernels according to the irreducible angular momentum representations. The same has to be done for the solution of (9.21). This prevents to come to such a simple eigenvalue equation like (9.24) in nonlinear spinor theory. Because after expansion with angular momentum representations (9.21) becomes a genuine nonlocal equation and the eigenvalues cannot be calculated by (9.24). But still after this expansion FREDHOLM theory is applicable for numerical computation. So we see, that nonlocal graphs in a theory with local interaction play a role due to rotational invariance. If this rotational invariance does not occur like for the anharmonic oscillator the eigenvalues can be determined from local graphs only as given by (9.24). The question of angular momentum representation of kernels is discussed in detail in a paper of DÜRR and WAGNER [15], but here it is not in the range of our investigation.

[15] H. P. DÜRR and F. WAGNER, Max-Planck-Institut für Physik und Astrophysik, Munich, preprint [1967].

## 10. Construction of the Partial Resolvents

We finally discuss the explicit construction of the partial resolvents used in the previous section. We first consider the problem of the construction of $\Gamma_{\lambda_1}$. According to (9.10) it is the resolvent of an equation of the following type

$$\psi(q_3 | q_1 q_2) = 6 f_1(q_3) \int k(q_{\lambda_1}, q_3 - \eta) \psi(q_{\lambda_1} + q_3 - \eta | \eta q_{\lambda_2}) \mathrm{d}\eta + f(q_3; q_1 q_2). \tag{10.1}$$

Now we substitute $q_{\lambda_1} = \alpha$ and $q_3 = \vartheta - \alpha$ in (10.1) where according to (8.15) we have $\vartheta = \omega - q_{\lambda_2}$. Because $\vartheta$ appears only as a parameter, we assume that $\vartheta$ has an arbitrary but fixed value, and define

$$\psi(\vartheta - \alpha | \alpha q_{\lambda_2}) := \chi(\alpha), \quad 6 f_1(\vartheta - \alpha) k(\alpha, \vartheta - \alpha - \eta) := M(\alpha, \eta; \vartheta), \quad f(\vartheta - \alpha; \alpha q_{\lambda_2}) := g(\alpha). \tag{10.2}$$

Then Eq. (10.1) can be written

$$\chi(\alpha) = \int M(\alpha, \eta; \vartheta) \chi(\eta) \mathrm{d}\eta + g(\alpha). \tag{10.3}$$

The kernel $M(\alpha\eta; \vartheta)$ is explicitly given by

$$M(\alpha\eta; \vartheta) = \frac{6}{2\pi} \left[4 - \frac{6}{2\pi} \int \tilde{G}'(\varrho) \tilde{F}(\vartheta - \alpha - \varrho) \mathrm{d}\varrho \right]^{-1} \cdot [\tilde{G}'(\alpha) \tilde{F}(\vartheta - \alpha - \eta) + \tilde{F}(\alpha) \tilde{G}'(\vartheta - \alpha - \eta)] \tag{10.4}$$

according to (10.2), (6.8) and (9.6). The functions $\tilde{G}$ and $\tilde{F}$ follow from Appendix V. We observe that the kernel $M(\alpha\eta; \vartheta)$ is an analytic function of the parameter $\vartheta$ and is built up by functions of the type

$$f(\alpha; c_k) =: [\alpha^2 - c_k^2 + i\varepsilon]^{-1} \quad (c_k > 0, \text{reel}). \tag{10.5}$$

These can be integrated only in the FEYNMANN sense; especially one can define a scalar product in the space of them by the definition

$$(f_1, f_2) =: \frac{1}{2\pi i} \int f(\alpha; c_1) \cdot f(\alpha; c_2) \mathrm{d}\alpha = [2 c_1 c_2 (c_1 + c_2)]^{-1} \tag{10.6}$$

showing all required properties of a scalar product. Calculating the norm of $M(\alpha\eta; \vartheta)$ with this prescription we get

$$(1/2\pi i)^2 \int M(\alpha\eta; \vartheta)^2 \mathrm{d}\alpha \, \mathrm{d}\eta = d(\vartheta) \tag{10.7}$$

where $d(\vartheta)$ is a meromorphic function of the parameter $\vartheta$ which is also built up by functions $f(\alpha; c_i)$. This means that the function $d(\vartheta)$ is bounded except in the surroundings of possible poles $\vartheta_i$. Thus we have some corresponding domains $B_i$ with

$$|d(\vartheta)| < K_i \quad \text{for} \quad \vartheta \in B_i \quad (K_i > 0). \tag{10.8}$$

Therefore we now can apply FREDHOLM theory[16] for the resolvent construction of (10.3) in these domains and obtain in this way

$$\chi(\alpha) = g(\alpha) + \int \Gamma_i(\alpha\eta; \vartheta) g(\eta) \mathrm{d}\eta. \tag{10.9}$$

For each domain $B_i$, the $\Gamma_i(\alpha\eta; \vartheta)$ are also square integrable in the same way as $M(\alpha\eta; \vartheta)$. Continuating $\Gamma_i(\alpha\eta; \vartheta)$ from one domain to the others we should get one single FREDHOLM resolvent $\Gamma(\alpha\eta; \vartheta)$ with

$$\Gamma(\alpha\eta; \vartheta) = R(\alpha\eta; \vartheta)/D(\vartheta) \tag{10.10}$$

where $R(\alpha\eta; \vartheta)$ is the FREDHOLM minor and $D(\vartheta)$ the FREDHOLMian determinant of $M(\alpha\eta; \vartheta)$. Further investigation on this subject is done in an other paper. Returning to the original variables the solution of (10.1) goes over into

$$\psi(q_3 | q_1 q_2) = f(q_3; q_1 q_2) + \int \Gamma(q_3; q_{\lambda_1} \eta) f(q_3 + q_{\lambda_1} - \eta; \eta q_{\lambda_2}) \mathrm{d}\eta. \tag{10.11}$$

[16] W. SMIRNOW, Lehrgang der höheren Mathematik IV, VEB Verlag der Wissenschaften, Berlin 1958, Chapt. I, 34.

Now we turn to the construction of $\Lambda$, being the resolvent of an equation of the following type

$$\psi(q_3 | q_1 q_2) = \int \Lambda(q_3; q_1 q_2; \xi \eta)\, \psi(q_1 + q_2 + q_3 - \xi - \eta | \xi \eta)\, d\xi\, d\eta + f(q_3 | q_1 q_2). \tag{10.12}$$

As before we substitute $q_1 = \alpha$, $q_2 = \beta$, $q_3 = \vartheta - \alpha - \beta$ in (10.12), where according to (8.6) we have $\vartheta = \omega$. Defining

$$\psi(\vartheta - \alpha - \beta | \alpha\beta) =: \chi(\alpha\beta), \quad \Lambda(\vartheta - \alpha - \beta | \alpha\beta; \xi\eta) =: M(\alpha\beta; \xi\eta; \vartheta), \quad f(\vartheta - \alpha - \beta | \alpha\beta) =: g(\alpha, \beta) \tag{10.13}$$

Eq. (10.12) can be written $\chi(\alpha\beta) = \int M(\alpha\beta; \xi\eta; \vartheta)\, \chi(\xi\eta)\, d\xi\, d\eta + g(\alpha\beta)$. (10.14)

Again we have to discuss the properties of the kernel $M(\alpha\beta; \xi\eta; \vartheta)$ which reads explicitly

$$\begin{aligned} M(\alpha\beta; \xi\eta; \vartheta) = {} & 36 f_1(\vartheta - \alpha - \beta) f_2(\alpha + \beta; \vartheta - \alpha - \beta)\, k(\alpha\beta)\, k(\alpha + \beta - \xi, \xi - \eta) \\ & + 6 \operatorname*{Sym}_{\alpha\beta} \Gamma(\vartheta - \alpha - \beta; \alpha, \xi) f_1(\vartheta - \beta - \xi)\, k(\beta, \vartheta - \beta - \xi - \eta) \\ & + 36 \operatorname*{Sym}_{\alpha\beta} \int \Gamma(\vartheta - \alpha - \beta, \alpha, \vartheta - \beta - \varrho) f_1(\varrho) f_2(\vartheta - \varrho, \varrho) \\ & \times k(\beta, \vartheta - \beta - \varrho)\, k(\vartheta - \xi - \varrho, \varrho - \eta)\, d\varrho. \end{aligned} \tag{10.15}$$

Like in the case of $\Gamma$, $M(\alpha\beta; \xi\eta; \vartheta)$ is built up by functions $f(\alpha; c_k)$. Keeping in mind the definition of $f_1(q)$, $f_2(q_1, q_2)$ and $k(q_1, q_2)$ and that $\Gamma(q_3; q_1 q_2)$ has also to be of that type we can use the same arguments as before to construct the FREDHOLMian resolvent of (10.14). So we obtain

$$\chi(\alpha\beta) = g(\alpha\beta) + \int \Lambda(\alpha\beta; \xi\eta; \vartheta)\, g(\xi\eta)\, d\xi\, d\eta. \tag{10.16}$$

$\Lambda$ having the form as in (10.10). Returning again to the original variables (10.16) results in

$$\psi(q_3 | q_1 q_2) = f(q_3 | q_1 q_2) + \int \Lambda(q_3; q_1 q_2; \xi\eta)\, f(q_1 + q_2 + q_3 - \xi - \eta | \xi\eta)\, d\xi\, d\eta. \tag{10.17}$$

The resolvent $\Pi$ is constructed in the same way as the resolvents $\Gamma$ and $\Lambda$. Therefore only considerations about the integrability of the kernel are necessary. They run on the same lines as before, because the kernel is also composed by functions of type (10.5).

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## Appendix I

We demonstrate the differentiation of a time-ordered product. It is most instructive to start with the $T$-product of $q(t_1) \dots q(t_n)\, p(t)$ which can be written

$$\begin{aligned} T q(t_1) \dots q(t_n)\, p(t) = P \sum_{\lambda_1 \dots \lambda_n} [\, & p(t)\, q(t_{\lambda_1}) \dots q(t_{\lambda_n})\, \Theta(t - t_{\lambda_1})\, \Theta(t_{\lambda_1} - t_{\lambda_2}) \dots \Theta(t_{\lambda_{n-1}} - t_{\lambda_n}) \\ & + \sum_{k=1}^{n-1} q(t_{\lambda_1}) \dots q(t_{\lambda_k})\, p(t)\, q(t_{\lambda_{k+1}}) \dots q(t_{\lambda_n})\, \Theta(t_{\lambda_1} - t_{\lambda_2}) \dots \Theta(t_{\lambda_k} - t)\, \Theta(t - t_{\lambda_{k+1}}) \dots \Theta(t_{\lambda_{n-1}} - t_{\lambda_n}) \\ & + q(t_{\lambda_1}) \dots q(t_{\lambda_k})\, p(t)\, \Theta(t_{\lambda_1} - t_{\lambda_2}) \dots \Theta(t_{\lambda_n} - t)]. \end{aligned} \tag{I.1}$$

Using the auxiliary formulas

$$\Theta(t_{\lambda_1} - t)\, \Theta(t - t_{\lambda_2}) = \Theta(t_{\lambda_1} - t_{\lambda_2})\, [\vartheta(t - t_{\lambda_2}) - \vartheta(t - t_{\lambda_1})], \tag{I.2}$$

$$\vartheta(t - t_{\lambda_1}) := \tfrac{1}{2} [\Theta(t - t_{\lambda_1}) - \Theta(t_{\lambda_1} - t)] \tag{I.3}$$

we obtain from (I.1)

$$\begin{aligned} T q(t_1)\dots q(t_n)\,p(t) = P \sum_{\lambda_1\dots\lambda_n} \prod_{i=1}^{n-1} \Theta(t_{\lambda_i} - t_{\lambda_{i+1}}) [\,& p(t)\,q(t_{\lambda_1})\dots q(t_{\lambda_n})\,\Theta(t - t_{\lambda_1}) \\ & + \sum_{k=1}^{n-1} q(t_{\lambda_1})\dots q(t_{\lambda_k})\,p(t)\,q(t_{\lambda_{k+1}})\dots q(t_{\lambda_n})\,\vartheta(t - t_{\lambda_{k+1}}) \\ & - \sum_{k=1}^{n-1} q(t_{\lambda_1})\dots q(t_{\lambda_k})\,p(t)\,q(t_{\lambda_{k+1}})\dots q(t_{\lambda_n})\,\vartheta(t - t_{\lambda_k}) + q(t_{\lambda_1})\dots q(t_{\lambda_k})\,p(t)\,\Theta(t_{\lambda_n} - t)] \\ = P \sum_{\lambda_1..\lambda_n} \prod_{i=1}^{n-1} \Theta(t_{\lambda_i} - t_{\lambda_{i+1}}) \{ & [\Theta(t - t_{\lambda_1})\,p(t)\,q(t_{\lambda_1}) - \vartheta(t - t_{\lambda_1})\,q(t_{\lambda_1})\,p(t)]\,q(t_{\lambda_2})\dots q(t_{\lambda_k}) \\ & + \sum_{k=2}^{n-1} \vartheta(t - t_{\lambda_k})\,q(t_{\lambda_1})\dots q(t_{\lambda_{k-1}})\,[p(t), q(t_{\lambda_k})]\,q(t_{\lambda_{k+1}})\dots q(t_{\lambda_k}) \\ & + q(t_{\lambda_1})\dots q(t_{\lambda_{k-1}})\,[\vartheta(t - t_{\lambda_n})\,p(t)\,q(t_{\lambda_n}) + \Theta(t_{\lambda_n} - t)\,q(t_{\lambda_n})\,p(t)]\} \end{aligned} \tag{I.4}$$

from this follows by differentiation observing (1.1) and (I.3)

$$\begin{aligned} \frac{\partial}{\partial t} T q(t_1)\dots q(t_n)\,p(t) = & - T q(t_1)\dots q(t_n)\,q^3(t) \\ & - i P \sum_{\lambda_1..\lambda_n} \prod_{i=1}^{n-1} \Theta(t_{\lambda_i} - t_{\lambda_{i+1}}) \sum_{k=1}^{n} \delta(t - t_{\lambda_k})\,q(t_{\lambda_1})\dots q(t_{\lambda_{k-1}})\,e(t_{\lambda_k})\,q(t_{\lambda_{k+1}})\dots q(t_{\lambda_n}) \end{aligned} \tag{I.5}$$

with

$$[p(t), q(t)]_- := -i\,e(t)\,. \tag{I.6}$$

Because of the $\delta$-function $t_{\lambda_k}$ can be substituted by $t$ and $e(t)$ is the unity matrix. Thus the last term in (I.5) can be written

$$\begin{aligned} & - i \sum_{r=1}^{n} \delta(t - t_r) [\sum_{k=1}^{n} P \sum_{\substack{\lambda_1\dots\lambda_{k-1}\lambda_{k+1}\dots\lambda_n \\ \neq r}} \prod_{i=1}^{n-1} \Theta(t_{\lambda_i} - t_{\lambda_{i+1}})\,q(t_{\lambda_1})\dots q(t_{\lambda_{k-1}})\,q(t_{\lambda_{k+1}})\dots q(t_{\lambda_n})]/_{t_{\lambda k} = t_{\lambda r}} \\ = & - i \sum_{r=1}^{n} \delta(t - t_r)\,P \sum_{\substack{\lambda_1\dots\lambda_{n-1} \\ \neq r}} q(t_{\lambda_1})\dots q(t_{\lambda_{n-1}})\,[\Theta(t_r - t_{\lambda_1})\dots\Theta(t_{\lambda_{n-2}} - t_{\lambda_{n-1}}) \\ & + \Theta(t_{\lambda_1} - t_r)\,\Theta(t_r - t_{\lambda_2})\dots\Theta(t_{\lambda_{n-2}} - t_{\lambda_{n-1}}) + \cdots + \Theta(t_{\lambda_1} - t_{\lambda_2})\dots\Theta(t_{\lambda_{n-1}} - t_r)] \\ = & - i \sum_{r=1}^{n} \delta(t - t_r)\,P \sum_{\substack{\lambda_1\dots\lambda_{n-1} \\ \neq r}} \prod_{i=1}^{n-2} \Theta(t_{\lambda_i} - t_{\lambda_{i+1}})\,q(t_{\lambda_1})\dots q(t_{\lambda_{n-1}}) \\ & \times [\Theta(t_r - t_{\lambda_1}) - \vartheta(t_r - t_{\lambda_1}) + \vartheta(t_r - t_{\lambda_{n-1}}) + \Theta(t_{\lambda_{n-1}} - t_r)] \\ = & \; i \sum_{r=1}^{n} \delta(t - t_r)\,P \sum_{\substack{\lambda_1\dots\lambda_{n-1} \\ \neq r}} q(t_{\lambda_1})\dots q(t_{\lambda_{n-1}}) \prod_{i=1}^{n-2} \Theta(t_{\lambda_i} - t_{\lambda_{i+1}})\,. \end{aligned} \tag{I.7}$$

Therefore we finally obtain

$$\frac{\partial}{\partial t} T q(t_1)\dots q(t_n)\,p(t) = - T q(t_1)\dots q(t_n)\,q^3(t) - i \sum_{r=1}^{n} \delta(t - t_r)\, T q(t_1)\dots q(t_{r-1})\,q(t_{r+1})\dots q(t_n)\,. \tag{I. 8}$$

Replacing in (I.1) $p(t)$ by $q(t)$ and performing the same operations we further obtain

$$\frac{\partial}{\partial t} T q(t_1)\dots q(t_n)\,q(t) = T q(t_1)\dots q(t_n)\,p(t)\,. \tag{I.9}$$

Now we use both formulae for the derivation of the functional equation. We have

$$\frac{\delta}{\delta j(t)} \mathfrak{T}_\varrho(j) = \sum_{n=0}^{\infty} \frac{(i)^{n+1}}{n!} \int \langle 0 | T q(t)\,q(\xi_1)\dots q(\xi_n) | \Psi_\varrho \rangle\, j(\xi_1)\dots j(\xi_n)\,d\xi_1\dots d\xi_n\,, \tag{I.10}$$

$$\frac{d}{dt} \frac{\delta}{\delta j(t)} \mathfrak{T}_\varrho(j) = \sum_{n=0}^{\infty} \frac{(i)^{n+1}}{n!} \int \langle 0 | T p(t)\,q(\xi_1)\dots q(\xi_n) | \Psi_\varrho \rangle\, j(\xi_1)\dots j(\xi_n)\,d\xi_1\dots d\xi_n \tag{I.11}$$

and by further differentiation and substitution of (I.8) into (I.11)

$$\frac{d^2}{dt^2}\frac{\delta}{\delta j(t)}\mathfrak{T}_\varrho(j) = \sum_{n=0}^{\infty}\frac{(i)^n}{n!}(i)^3\int \langle 0|T q^3(t)\, q(\xi_1)\dots q(\xi_n)|\Psi_\varrho\rangle j(\xi_1)\dots j(\xi_n)\,\mathrm{d}\xi_1\dots\mathrm{d}\xi_n \tag{I.12}$$

$$-\sum_{n=0}^{\infty}\frac{(i)^{n+1}}{n!}\sum_{r=1}^{n}\int \langle 0|T q(\xi_1)\dots q(\xi_{r-1})\, q(\xi_{r+1})\dots q(\xi_n)|\Psi_\varrho\rangle j(\xi_1)\dots j(\xi_{r-1})\, j(t)\, j(\xi_{r+1})\dots j(\xi_n)\mathrm{d}\xi_1 \underset{\neq r}{\dots} \mathrm{d}\xi_n$$

changing the integration variables in (I.12) we finally obtain

$$\frac{d^2}{dt^2}\frac{\delta}{\delta j(t)}\mathfrak{T}_\varrho(j) = \frac{\delta^3}{\delta j(t)^3}\mathfrak{T}_\varrho(j) + i\, j(t)\,\mathfrak{T}_\varrho(j)\,. \tag{I.13}$$

## Appendix II

Here we discuss the explicit calculation of the operator representations (6.2) and (6.3). First we remember that all DYSON base functionals can be written in the form

$$D_k(j) = F_k(j)\exp\{-\tfrac{1}{2}\textstyle\int j(\xi)\, F(\xi-\eta)\, j(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\} \tag{II.1}$$

whereas the reciprocal base functionals are given by

$$D^l(j) \equiv S_l(j) = F^l(j)\exp\{\tfrac{1}{2}\textstyle\int j(\xi)\, F(\xi-\eta)\, j(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\} \tag{II.2}$$

with $F^l(j) \equiv R_l(j)$ defined by (4.3) and (4.4). Then we have by inserting (II.1) and (II.2) into (6.2) and (6.3)

$$A_i^k = \textstyle\int F^k(j)\, j(t)\left[\frac{\delta}{\delta j(t)} - I(t)\right]\mathrm{d}t\, F_i(j)\,\mathrm{d}j \tag{II.3}$$

and

$$C_i^k = \textstyle\int F^k(j)\, j(t)\, G(t-t')\, n\left((j(t'), \frac{\delta}{\delta j(t')}\right)\mathrm{d}t\,\mathrm{d}t'\, F_i(j)\,\mathrm{d}j \tag{II.4}$$

with

$$I(t) := \textstyle\int F(t-\eta)\, j(\eta)\,\mathrm{d}\eta \tag{II.5}$$

and

$$n\left(j(t), \frac{\delta}{\delta j(t)}\right) := i\, j(t) + \frac{\delta^3}{\delta j(t)^2} - 3\, I(t)\frac{\delta^3}{\delta j(t)^3} + 3\, I(t)^2\frac{\delta}{\delta j(t)} - I(t)^3\,. \tag{II.6}$$

Now the representations (II.3) and (II.4) can be directly evaluated by means of the following auxiliary recursion formulas. We define

$$\mathfrak{P}^m(t_1\dots t_m) := \frac{1}{i^m}\frac{\delta^m}{\delta j(t_1)\dots\delta j(t_m)}\,. \tag{II.7}$$

Then we investigate the effect of the application of $\mathfrak{P}^m$ on certain functionals.

$$\mathfrak{P}^m(t_1\dots t_m)\sum_{n=1}^{\infty}\frac{i^n}{n!}\int g(\xi_{n+1})\, f_n(\xi_1\dots\xi_n)\, j(\xi_1)\dots j(\xi_{n+1})\,\mathrm{d}\xi_1\dots\mathrm{d}\xi_{n+1}\big|_{j=0}$$

$$= \mathfrak{P}^{m-1}(t_1\dots t_{m-1})\sum_n\frac{i^{n-1}}{n!}\int g(t_m)\, f_m(\xi_1\dots\xi_n)\, j(t_1)\dots j(t_n)\, d\xi_1\dots\mathrm{d}\xi_n\big|_{j=0} \tag{II.8}$$

$$+ \mathfrak{P}^{m-1}(t_1\dots t_{m-1})\sum_n\frac{i^{n-1}}{n!}\int g(\xi_{n+1})\sum_{j=1}^{n} f_n(\xi_1\dots\xi_n)\,\delta(t_m-\xi_j)\, j(\xi_1)\dots j(\xi_{j-1})\, j(\xi_{j+1})\dots j(\xi_{n+1})$$

$$\times\, \mathrm{d}\xi_1\dots\mathrm{d}\xi_{n+1}\big|_{j=0}$$

$$= (1/i)\, g(t_m)\, f_{m-1}(t_1\dots t_{m-1})$$

$$+ \mathfrak{P}^{m-1}(t_1\dots t_{m-1})\sum_n\frac{i^{n-1}}{(n-1)!}\int g(\xi_n)\, f_n(\xi_1\dots\xi_{n-1}, t_m)\, j(\xi_1)\dots j(\xi_n)\,\mathrm{d}\xi_1\dots\mathrm{d}\xi_n\big|_{j=0}\,.$$

Repeated application of this recursion formula finally leads to

$$\mathfrak{P}^m(t_1\dots t_m)\sum_{n=1}^{\infty}\frac{i^n}{n!}\int g(\xi_{n+1})\, f_n(\xi_1\dots\xi_n)\, j(\xi_1)\dots j(\xi_{n+1})\,\mathrm{d}\xi_1\dots\mathrm{d}\xi_{n+1}\big|_{j=0} \tag{II.9}$$

$$= \sum_{\lambda_1=1}^{m}\frac{1}{i}\, g(t_{\lambda_1})\, f_{m-1}(t_{\lambda_2}\dots t_{\lambda_m})\,.$$

The derivation of the further recursion formulas runs completely analogous. We obtain

$$\mathfrak{P}^m(t_1 \dots t_m) \sum_{n=1}^{\infty} \frac{i^n}{n!} \int g(\xi_{n+1} \dots \xi_{n+\nu}) f_n(\xi_1 \dots \xi_n) j(\xi_1) \dots j(\xi_{n+\nu}) \, d\xi_1 \dots d\xi_{n+\nu} \big|_{j=0}$$

$$= P \sum_{\lambda_1 \dots \lambda_\nu = 1}^{m} g(t_{\lambda_1} \dots t_{\lambda_\nu}) f_{m-\nu}(t_{\lambda_{\nu+1}} \dots t_{\lambda_m}) \cdot \frac{1}{i^\nu} \tag{II.10}$$

where the right-hand side disappears if $m-\nu$ is negative.
Now in order to apply the recursion formulas (II.10) we represent the base functionals $F_i(j)$ by

$$F_i(j) = (i^r/r!) \int \delta(t_1' - \xi_1) \dots \delta(t_r' - \xi_r) j(\xi_1) \dots j(\xi_r) \, d\xi_1 \dots d\xi_r . \tag{II.11}$$

Then the operator representation can easily be given by means of (II.10). We examplify this for the operator (II.3). For the first term we have

$$\int F^k(j) j(t) \frac{\delta}{\delta j(t)} dt \, F_r(j) \, dj \equiv$$

$$\mathfrak{P}^k(t_1 \dots t_k) \int j(t) \frac{\delta}{\delta j(t)} \frac{i^r}{r!} \delta(t_1' - \xi_1) \dots \delta(t_r' - \xi_r) j(\xi_1) \dots j(\xi_r) \, d\xi_1 \dots d\xi_r \, dt \big|_{j=0}$$

$$= \mathfrak{P}^k(t_1 \dots t_k) \int j(t) \frac{r i^r}{r!} \delta(t_1' - \xi_1) \dots \delta(t_{r-1} - \xi_{r-1}) \delta(t_r - t) j(\xi_1) \dots j(\xi_{r-1}) \, d\xi_1 \dots d\xi_{r-1} \, dt \big|_{j=0}$$

$$= \mathfrak{P}^k(t_1 \dots t_k) \int \frac{i^r}{r!} r \, \delta(t_1' - \xi_1) \dots \delta(t_r' - \xi_r) j(\xi_1) \dots j(\xi_r) \, d\xi_1 \dots d\xi_r \big|_{j=0} \tag{II.12}$$

$$= r \, \delta(t_1' - t_1) \dots \delta(t_r' - t_r) \, \delta_{kr} ,$$

for the second term we have

$$\int F^k(j) j(t) I(t) \, dt \, F_r(j) \, dj$$

$$\equiv \mathfrak{P}^k(t_1 \dots t_k) \int j(t) F(t-\eta) j(\eta) \frac{i^r}{r!} \delta(t_1' - \xi_1) \dots \delta(t_r' - \xi_r) j(\xi_1) \dots j(\xi_r) \, d\xi_1 \dots d\xi_r \, dt \, d\eta \big|_{j=0} \tag{II.13}$$

$$= \mathfrak{P}^k(t_1 \dots t_k) \int \frac{i^r}{r!} F(\xi_{r+1} - \xi_{r+2}) \delta(t_1' - \xi_1) \dots \delta(t_r' - \xi_r) j(\xi_1) \dots j(\xi_{r+2}) \, d\xi_1 \dots d\xi_{r+2} \big|_{j=0}$$

$$= -\sum_{\lambda_1 \lambda_2 = 1}^{k} F(t_{\lambda_1} - t_{\lambda_2}) \delta(t_1' - t_{\lambda_3}) \dots \delta(t_{k-2}' - t_{\lambda_k}) \, \delta_{k-2,r}$$

by means of the auxiliary formula (II.10). Now we can easily write down the effect of the operator $A_r^k$ on $\varphi_r(\varrho)$. We obtain by observing the definitions of Sect. 5

$$A_r^k \varphi^r(\varrho) \equiv \sum_r \int A_r^k(t_1 \dots t_k, t_1' \dots t_r') \varphi_r(t_1' \dots t_r') \, dt_1' \dots dt_r'$$

$$= k \, \varphi_k(t_1 \dots t_k) - \sum_{\lambda_1 \lambda_2 = 1}^{k} F(t_{\lambda_1} - t_{\lambda_2}) \varphi_{k-2}(t_{\lambda_3} \dots t_{\lambda_k}) . \tag{II.14}$$

The calculation of $C_r{}^k$ runs completely analogous to that of $A_r{}^k$. Therefore it is not necessary to discuss this separately.

## Appendix III

In this appendix the construction formulas of Sect. 7 are discussed explicitly. We define the set of variables

$$l_k^m := q_{k+1} \dots q_{2m-k} . \tag{III.1}$$

We then have especially for $k=0$ the original set

$$l_0^m := q_1 \dots q_{2m} . \tag{III.2}$$

Furthermore in all formulas we substitute instead of $q_\lambda$ simply $\lambda$ for abbreviation. Then by definition (7.4) the contraction for a symmetrical function $s(l_0)$ becomes

$$P_k \dots P_1 s_{2m}(l_0^m) := \int s_{2m}(\xi_1 \dots \xi_k, q_1 - \xi_1, \dots q_k - \xi_k, l_k^m) \, d\xi_1 \dots d\xi_k \qquad \text{(III.3)}$$
$$:= s_{2m}^k(l_k^m \,|\, 1 \dots k) .$$

When we define the sets further

$$l_{k,\lambda_1 \dots \lambda_i}^m := q_{k+1} \dots q_{k+\lambda_1-1} q_{k+\lambda_1+1} \dots q_{k+\lambda_i-1} q_{k+\lambda_i+1} \dots q_{2m-k} \qquad \text{(III.4)}$$
$$\lambda_1 \dots \lambda_i := q_{\lambda_1} \dots q_{\lambda_i}$$

the formulas to be contracted can be written as

$$L_i^0(m) := \sum_{(\lambda_1 \dots \lambda_i \in l_0^m)} g(\lambda_1 \dots \lambda_i) \, s_{2m-i}(l_{0,\lambda_1 \dots \lambda_i}^m) \qquad \text{(III.5)}$$

where $g(\lambda_1 \dots \lambda_i)$ is a symmetrical function in all arguments. In the dynamical equations (7.5) $i=1,\dots,4$ occurs. It is convenient to define further the following expressions

$$L_i^k(m) := \sum_{(\lambda_1 \dots \lambda_i \in l_k^m)} g(\lambda_1 \dots \lambda_i) \, s_{2m-i}^k(l_{k\lambda_1 \dots \lambda_i}^m \,|\, 1 \dots k) . \qquad \text{(III.6)}$$

Now we are ready to start the investigation of the recursion formulae for contraction. We first consider the case $i=1$. We have

$$L_1^0(m) = \sum_{\lambda_1=1}^{2m} g(\lambda_1) \, s_{2m-1}(l_{0\lambda_1}^m) \qquad \text{(III.7)}$$
$$= g(1) \, s_{2m-1}(l_{1,2m}^m) + g(2m) \, s_{2m-1}(l_1^m, 1) + \sum_{\lambda_1 \in l_1^m} g(\lambda_1) \, s_{2m-1}(l_{1,\lambda_1;1,2m}^m)$$

and applying $P_1$ to (III.7) we obtain

$$P_1 L_1^0(m) = 2 \int g(1-\xi) \, s_{2m-1}(l_1^m \xi) \, d\xi + L_1^1(m) . \qquad \text{(III.8)}$$

Continuing in the same way by dividing $L_1^1(m)$ into the corresponding terms to (III.7) we obtain finally

$$P_k \dots P_1 L_1{}^0(m) = 2 \sum_{\mu_1=1}^{k} \int g(\mu_1 - \xi) \, s_{2m-i}^{k-1}(l_k^m \xi \,|\, 1 \dots k \mp \mu_1) \, d\xi + L_1^k(m) . \qquad \text{(III.9)}$$

Now we consider the case $i=2$. There we have

$$L_2^0(m) = \sum_{(\lambda_1,\lambda_2)=1}^{2m} g(\lambda_1 \lambda_2) \, s_{2m-2}(l_{0\lambda_1\lambda_2}^m) = g(1, 2m) \, s_{2m-2}(l_1^m) + \sum_{\lambda_1 \in l_1^m} g(1, \lambda_1) \, s_{2m-2}(l_{1\lambda_1}^m, 2m)$$
$$+ \sum_{\lambda_1 \in l_1^m} g(2m, \lambda_1) \, s_{2m-2}(l_{1\lambda_1}^m, 1) + \sum_{(\lambda_1,\lambda_2) \in l_1^m} g(\lambda_1, \lambda_2) \, s_{2m-2}(l_{1,\lambda_1\lambda_2}^m; 1, 2m) . \qquad \text{(III.10)}$$

Applying $P_1$ to (III.10) we obtain

$$P_1 L_2^0(m) = g^1(1) \, s_{2m-2}(l_1^m) + 2 \sum_{\lambda_1 \in l_1^m} \int g(1-\xi, \lambda_1) \, s_{2m-2}(l_{1,\lambda_1}^m \xi) \, d\xi + L_2^1 \qquad \text{(III.11)}$$

and finally

$$P_k \dots P_1 L_2^0(m) = \sum_{g^1(\mu_1)}^{k} g^1(\mu_1) \, s_{2m-2}^{k-1}(l_k^m \,|\, 1 \dots k \mp \mu_1)$$
$$+ 2 \sum_{\lambda_1 \in l_k^m} \sum_{\mu_1=1}^{k} \int g(\mu_1 - \xi, \lambda_1) \, s_{2m-2}^{k-1}(l_{k,\lambda_1}^m; \xi \,|\, 1 \dots k \mp \mu_1) \, d\xi \qquad \text{(III.12)}$$
$$+ 4 \sum_{(\mu_1\mu_2)=1}^{k} \int g(\mu_1 - \xi, \mu_2 - \eta) \, s_{2m-2}^{k-2}(l_k^m; \xi, \eta \,|\, 1 \dots k \mp \mu_1 \mu_2) \, d\xi \, d\eta + L_2^k(m) .$$

In the same way we obtain for $i=3$

$$\begin{aligned} P_k \dots P_1 L_3^0(m) = \sum_{\mu_1=1}^{k} \Big[ & \sum_{\lambda_1 \in l_k^m} g^1(\lambda_1 \mid \mu_1)\, s_{2m-3}^{k-1}\,(l_{k,\lambda_1}^m \mid 1 \dots k + \mu_1) \\ & + 2 \sum_{(\lambda_1 \lambda_2) \in l_k^m} \int g(\mu_1 - \eta, \lambda_1 \lambda_2)\, s_{2m-3}^{k-1}(l_{k,\lambda_1\lambda_2}^m, \eta \mid 1 \dots k + \mu_1)\, \mathrm{d}\eta \Big] \\ & + 2 \sum_{(\mu_1\mu_2)=1}^{k} \Big[ \int \operatorname{sym}_{\mu_1 \mu_2} g^1(\mu_1 - \xi \mid \mu_2)\, s_{2m-3}^{k-2}(l_k^m\ , \xi \mid 1 \dots k + \mu_1 \mu_2)\, \mathrm{d}\xi \\ & + 2 \sum_{\lambda_1 \in l_k^m} \int g(\mu_1 - \xi, \mu_2 - \eta, \lambda_1)\, s_{2m-3}^{k-2}(l_{k,\lambda_1}^m; \xi \eta \mid 1 \dots k + \mu_1 \mu_2)\, \mathrm{d}\xi\, \mathrm{d}\eta \Big] \end{aligned} \quad \text{(III.13)}$$

$$+ 8 \sum_{(\mu_1\mu_2\mu_3)=1}^{k} \int g(\mu_1 - \xi, \mu_2 - \eta, \mu_3 - \varrho)\, s_{2m-3}^{k-3}(l_k^m\ ; \xi \eta \varrho \mid 1 \dots k + \mu_1 \mu_2 \mu_3)\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\varrho + L_3{}^k(m)$$

and for $i = 4$ we get

$$\begin{aligned} P_k \dots P_1 L_4^0(m) = \sum_{\mu_1=1}^{k} \Big[ & \sum_{(\lambda_1\lambda_2) \in l_k^m} g^1(\lambda_1 \lambda_2 \mid \mu_1)\, s_{2m-4}^{k-1}(l_{k,\lambda_1\lambda_2}^m \mid 1 \dots k + \mu_1) \\ & + 2 \sum_{(\lambda_1\lambda_2\lambda_3) \in l_k^m} \int g(\mu_1 - \xi, \lambda_1 \lambda_2 \lambda_3)\, s_{2m-4}^{k-1}(l_{k,\lambda_1\lambda_2\lambda_3}^m; \xi \mid 1 \dots k + \mu_1)\, \mathrm{d}\xi \Big] \\ & + \sum_{(\mu_1\mu_2)=1}^{k} \Big[ g^2(\mu_1, \mu_2)\, s_{2m-4}^{k-2}(l_k^m \mid 1 \dots k + \mu_1 \mu_2) \\ & + 2 \sum_{\lambda_1 \in l_k^m} \int \operatorname{sym}_{\mu_1\mu_2} g^1(\mu_1 - \xi, \lambda_1 \mid \mu_2)\, s_{2m-4}^{k-2}(l_{k,\lambda_1}^m; \xi \mid 1 \dots k + \mu_1 \mu_2)\, \mathrm{d}\xi \qquad \text{(III.14)} \\ & + 4 \sum_{(\lambda_1\lambda_2) \in l_k^m} \int g(\mu_1 - \xi, \mu_2 - \eta, \lambda_1 \lambda_2)\, s_{2m-4}^{k-2}(l_{k,\lambda_1\lambda_2}^m; \xi \eta \mid 1 \dots k + \mu_1 \mu_2)\, \mathrm{d}\xi\, \mathrm{d}\eta \Big] \\ & + 4 \sum_{(\mu_1\mu_2\mu_3)=1}^{k} \Big[ \int \operatorname{sym}_{\mu_1\mu_2\mu_3} g^1(\mu_1 - \xi, \mu_2 - \eta \mid \mu_3)\, s_{2m-4}^{k-3}(l_k^m; \xi \eta \mid 1 \dots k + \mu_1 \mu_2 \mu_3)\, \mathrm{d}\xi\, \mathrm{d}\eta \\ & + 2 \sum_{\lambda_1 \in l_k^m} \int g(\mu_1 - \xi, \mu_2 - \eta, \mu_3 - \varrho, \lambda_1)\, s_{2m-4}^{k-3}; (l_{k,\lambda_1}^m; \xi \eta \varrho \mid 1 \dots k + \mu_1 \mu_2 \mu_3)\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\varrho \Big] \\ & + 16 \sum_{(\mu_1\mu_2\mu_3\mu_4)=1}^{k} \int g(\mu_1 - \xi, \mu_2 - \eta, \mu_3 - \varrho, \mu_4 - \tau)\, s_{2m-4}^{k-4}(l_k^m\, \xi \eta \varrho \tau \mid \\ & \qquad 1 \dots k + \mu_1 \mu_2 \mu_3 \mu_4)\, \mathrm{d}\xi\, \mathrm{d}\varrho\, \mathrm{d}\eta\, \mathrm{d}\tau + L_4{}^k(m)\,. \end{aligned}$$

Identifying for

$$i = 1 \qquad g(\lambda_1) \equiv g(\lambda_1)\,, \qquad s_{2m-1}(l_{0\lambda_1}^m) \equiv \int \varphi_{2m+2}^1(l_{0\lambda_1}^m; \eta \mid \lambda_1 - \eta)\, \mathrm{d}\eta\,, \quad \text{(III.15)}$$

$$i = 2 \qquad g(\lambda_1 \lambda_2) \equiv k(\lambda_1 \lambda_2)\,, \qquad s_{2m-2}(l_{0\lambda_1\lambda_2}^m) \equiv \varphi_{2m}^1(l_{0\lambda_1\lambda_2}^m \mid \lambda_1 + \lambda_2)\,, \quad \text{(III.16)}$$

or

$$g(\lambda_1 \lambda_2) \equiv h(\lambda_1)\, \delta(\lambda_1 + \lambda_2) \quad s_{2m-2}(l_{0\lambda_1\lambda_2}^m) \equiv \varphi_{2m-2}^0(l_{0\lambda_1\lambda_2}^m)\,, \quad \text{(III.17)}$$

$$i = 3 \quad g(\lambda_1 \lambda_2 \lambda_3) \equiv h_1(\lambda_1 \lambda_2 \lambda_3) \qquad s_{2m-3}(l_{0\lambda_1\lambda_2\lambda_3}^m) \equiv \varphi_{2m-2}^0(l_{0\lambda_1\lambda_2\lambda_3}^m, \lambda_1 + \lambda_2 + \lambda_3)\,, \quad \text{(III.18)}$$

and for

$$i = 4 \quad g(\lambda_1 \lambda_2 \lambda_3 \lambda_4) = h_2(\lambda_1 \lambda_2 \lambda_3 \lambda_4) \quad s_{2m-4}(l_{0\lambda_1\lambda_2\lambda_3\lambda_4}^m) = \varphi_{2m-4}^0(l_{0\lambda_1\lambda_2\lambda_3\lambda_4}^m) \quad \text{(III.19)}$$

we obtain the corresponding equations of Sect. 7.

## Appendix IV

We give here the recursion formulas for $S_1$ and $S_2$ appearing in (8.21)

$$S_1(q_1 q_2) := \tfrac{1}{2}[k(q_1 q_2) + \sum_{\lambda_1=1}^{2} g(q_{\lambda_2}) P_1(q_{\lambda_1} q_{\lambda_2}) + N_1(q_1 q_2)]\, d(q_1 q_2)\,,$$

$$S_2(q_1 q_2 \xi) := \tfrac{1}{2}[N_2(q_1 q_2 \xi) + \sum_{\lambda_1=1}^{2} g(q_{\lambda_2}) P_2(q_{\lambda_1} q_{\lambda_2} \xi)]\, d(q_1 q_2)\,, \tag{IV.1}$$

$$d(q_1 q_2) := [1 - \sum_{\lambda_1=1}^{2} g(q_{\lambda_2}) P_3(q_{\lambda_1} q_{\lambda_2})]^{-1}\,, \tag{IV.2}$$

$$\begin{aligned} N_1(q_1 q_2) := & \int B(q_1 q_2 \varrho, q_1 + q_2 - \varrho)\, [f_1(q_1 + q_2)\, h(\varrho) \\ & + C g(q_1 - \varrho, q_1 + q_2) + C g(\varrho, q_1 + q_2 - \varrho, 0)]\, \mathrm{d}\varrho \\ & + \int B(q_1 q_2 \varrho \varkappa)\, M_1(\varrho, \varkappa, q_1 + q_2 - \varrho - \varkappa)\, \mathrm{d}\varrho\, \mathrm{d}\varkappa\,, \end{aligned} \tag{IV.3}$$

$$\begin{aligned} N_2(q_1 q_2 \eta) := & B^1(q_1 + q_2 \,|\, q_1 q_2)\, C f_1(0) \\ & + \int M_2(\varrho, q_1 + q_2 - \varrho - \eta)\, [B(q_1 q_2 \eta \varrho) + B(q_1 q_2 \varrho \eta)]\, \mathrm{d}\varrho \\ & + \int M_3(\varrho, \varkappa, q_1 + q_2 - \varrho - \varkappa - \eta)\, B(q_1 q_2, \varrho, \varkappa)\, \mathrm{d}\varrho\, \mathrm{d}\varkappa\,, \end{aligned} \tag{IV.4}$$

$$P_1(q_1 q_2) := f_1(q_1 + q_2)\, h(q_1) + C g(q_1, -q_1, q_1 + q_2) + C g(q_1, q_2, 0) + M_1{}^1(q_2 \,|\, q_1)\,, \tag{IV.5}$$

$$P_2(q_1 q_2 \eta) := M_2(q_1 q_2 - \eta) + M_3^1(q \,|\, q_1 \eta)\,, \tag{IV.6}$$

$$P_3(q_1 q_2) := C f_1(0) + M_2^1(q_2)\,, \tag{IV.7}$$

$$C := \int h(\xi)\, \mathrm{d}\xi\,, \tag{IV.8}$$

$$\begin{aligned} M_1(q_1 q_2 q_3) := & \sum_{\lambda_1=1}^{2} [\Gamma(q_3; q_{\lambda_2}, -q_{\lambda_1})\, f_1(q_1 + q_2 + q_3)\, h(q_{\lambda_1}) \\ & + C\Gamma(q_3; q_{\lambda_2}, -q_{\lambda_1})\, g(q_{\lambda_1}, -q_{\lambda_1}, q_1 + q_2 + q_3) + C\Gamma(q_3; q_{\lambda_2}, q_3 + q_{\lambda_2})\, g(q_{\lambda_1}, q_3 + q_{\lambda_2}, 0)\,, \end{aligned} \tag{IV.9}$$

$$\begin{aligned} M_2(q_{\lambda_1} q_3) : \; = & A_1(q_{\lambda_1} q_3)\, f_1(q_3) + C\Gamma(q_3; q_{\lambda_1}, q_3 + q_{\lambda_1})\, f_1(0) \\ & + \int \Gamma(q_3; q_{\lambda_1} \xi)\, f_1(q_{\lambda_1} + q_3 - \xi)\, A_1(\xi, q_{\lambda_1} + q_3 - \xi)\, \mathrm{d}\xi\,, \end{aligned} \tag{IV.10}$$

$$\begin{aligned} M_3(q_1 q_2 q_3, \eta) : \; = & 6 f_1(q_3)\, h_1(q_1 q_2 q_3 - \eta) + g(q_1 q_2 q_3)\, A_2(q_1 + q_2, q_3, \eta) \\ & + 6 \sum_{\lambda_1=1}^{2} [[\int \Gamma(q_3; q_{\lambda_2} \xi)\, f_1(q_{\lambda_2} + q_3 - \xi)\, h_1(q_{\lambda_2}, \xi, q_{\lambda_2} + q_3 - \xi - \eta)\, \mathrm{d}\xi \\ & + \int \Gamma(q_3; q_{\lambda_2} \xi - q_{\lambda_1})\, g(q_{\lambda_1}, \xi - q_{\lambda_1}, q_1 + q_2 + q_3 - \xi)\, A_2(\xi, q_1 + q_2 + q_3 - \xi, \eta)\, \mathrm{d}\xi \\ & + \Gamma(q_3; q_{\lambda_2} \eta)\, f_1(q_{\lambda_2} + q_3 - \eta)\, A_1(q_{\lambda_1}, q_{\lambda_2} + q_3 - \eta)]\,, \end{aligned} \tag{IV.11}$$

$$A_1(q_1 q_2) := 2 h(q_1) + 3 h_1^1(q_1 \,|\, q_2)\,, \tag{IV.12}$$

$$A_2(q_1 q_2 \xi) := 2[2 h(q_1 - \xi) + 3 \sum_{\lambda_1=1}^{2} h_1^1(q_{\lambda_2} - \xi \,|\, q_{\lambda_1})]\,. \tag{IV.13}$$

All the other definitions are given in the preceding Sect. 9 especially the definitions of $h_1(q_1 q_2 q_3)$, $h(q)$ and $g(q)$ by (6.8), $f_1(q)$ and $f_2(q_1 q_2)$ by (9.6) and $g(q_1, q_2, q_3)$ by (9.8)

## Appendix V

In this appendix we discuss the singular functions $G$ and $F$ appearing in (6.8) and all derived equations.

Defining

$$G(t) = -\frac{1}{2\pi}\int \tilde{G}'(p)\, e^{-ipt}\, dp \qquad (V.1)$$

then from (3.3) follows for $G'$

$$\tilde{G}'(p) = [p^2 - 3F(0) + i\varepsilon]^{-1} \qquad (V.2)$$

where the small imaginary part effects FEYNMAN integration.

For the two point function $F$ we have the definition

$$F(t_1 - t_2) = \langle 0 | T q(t_1)\, q(t_2) | 0 \rangle \qquad (V.3)$$
$$= \sum_{n=1}^{\infty} |\langle 0 | q(0) | n \rangle|^2 e^{-i\omega_n|t_1 - t_2|}$$

with $\omega_n =: (E_n - E_0)$. The FOURIER transformed $\tilde{F}$

$$F(t) = \frac{1}{2\pi}\int \tilde{F}(p)\, e^{-ipt}\, dp \qquad (V.4)$$

then gives

$$\tilde{F}(p) = \sum_{n=1}^{\infty} |\langle 0 | q(0) | n \rangle|^2\, 2i\,\omega_n [p^2 - \omega_n^2 + i\varepsilon]^{-1}. \qquad (V.5)$$

$\tilde{F}$ itself can be calculated selfconsistently by the calculation of states of odd parity. We do show only the lowest approximation. In this approximation we put

$$\tilde{F}_{\text{app}}(p) = 2i\,|\langle 0 | q(0) | 1 \rangle|^2\, \omega_1 [p^2 - \omega_1^2 + i\varepsilon]^{-1} \qquad (V.6)$$

because we know from conventional quantum mechanics $|\langle 0 | q(0) | 1 \rangle| \gg |\langle 0 | q(0) | n \rangle|$ for $n = 2, \ldots, \infty$. From the commutation relation between $q(0)$ and $p(0)$ follows exactly the sum rule

$$\sum_{n=1}^{\infty} 2\,\omega_n |\langle 0 | q(0) | n \rangle|^2 = 1 \qquad (V.7)$$

and approximately

$$|\langle 0 | q(0) | 1 \rangle|^2\, 2\,\omega_1 = 1 \qquad (V.8)$$

and from this

$$\tilde{F}_{\text{app}}(p) = i[p^2 - \omega_1^2 + i\varepsilon]^{-1}. \qquad (V.9)$$

Further in this approximation follows from (V.3)

$$\tilde{F}_{\text{app}}(0) = |\langle 0 | q(0) | 1 \rangle|^2 = 1/2\,\omega_1. \qquad (V.10)$$

Therefore in this approximation we obtain

$$\tilde{G}'(p) = \left[p^2 - \frac{3}{2\omega_1} + i\varepsilon\right]^{-1} \qquad (V.11)$$

and we have to calculate selfconsistently only the root $\omega_1$ but no matrixelement. For this calculation we consider the lowest equation of states of odd parity of (7.2). It reads

$$\tilde{\varphi}_1(\omega) = \tilde{G}'(\omega)\int \varphi_3^1(\omega - \xi \,|\, \xi)\, d\xi \qquad (V.12)$$

if one assumes the center of gravity condition to be fulfilled for $\tilde{\varphi}_1$. From (V.12) follows

$$\left[\omega^2 - \frac{3}{2\,\omega_1}\right]\tilde{\varphi}_1(\omega) = \int \varphi_3^1(\omega - \xi \,|\, \xi)\, d\xi \qquad (V.13)$$

and assuming $\varphi_3^1$ and all higher $\varphi$-functions to vanish we have the root

$$\omega = \omega_1 = \sqrt[3]{\frac{3}{2}} = 1.1447. \qquad (V.14)$$

From this root all interesting constants can be calculated, for example $a =: \sqrt{3F(0)}$.